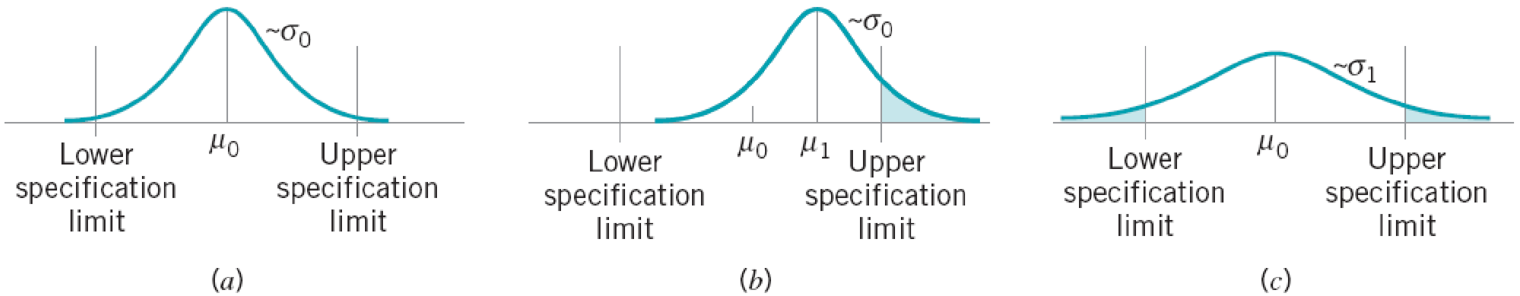
Control Charts for Variables



■ **FIGURE 6.1** The need for controlling both process mean and process variability. (*a*) Mean and standard deviation at nominal levels. (*b*) Process mean $\mu_1 > \mu_0$. (*c*) Process standard deviation $\sigma_1 > \sigma_0$.

6.2 Control Charts for $\overline{x}$ and R

6.2.1 Statistical Basis of the Charts

Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ, where both μ and σ are known. If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the average of this sample is

$$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

and we know that $\overline{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\overline{x}} = \sigma/\sqrt{n}$. Furthermore, the probability is $1 - \alpha$ that any sample mean will fall between

$$\mu + Z_{\alpha/2}\sigma_{\overline{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\overline{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \tag{6.1}$$

Therefore, if μ and σ are known, equation (6.1) could be used as upper and lower control limits on a control chart for sample means. As noted in Chapter 5, it is customary to replace $Z_{\alpha/2}$ by 3, so that three-sigma limits are employed. If a sample mean falls outside of these limits, it is an indication that the process mean is no longer equal to μ.

Subgroup Data with Unknown μ and σ

In practice, we usually will not know μ and σ. Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control. These estimates should usually be based on at least 20 to 25 samples. Suppose that m samples are available, each containing n observations on the quality characteristic. Typically, n will be small, often either 4, 5, or 6. These small sample sizes usually result from the construction of rational subgroups and from the fact that the sampling and inspection costs associated with variables measurements are usually relatively large. Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m$ be the average of each sample. Then the best estimator of μ, the process average, is the grand average—say,

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \tag{6.2}$$

Thus, $\bar{\bar{x}}$ would be used as the center line on the $\bar{x}$ chart.

To construct the control limits, we need an estimate of the standard deviation σ. Recall from Chapter 4 (Section 4.2) that we may estimate σ from either the standard deviations or the ranges of the m samples. For the present, we will use the range method. If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the range of the sample is the difference between the largest and smallest observations; that is,

$$R = x_{\max} - x_{\min}$$

Let $R_1, R_2, \ldots, R_m$ be the ranges of the m samples. The average range is

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \tag{6.3}$$

$$\hat{\sigma} = \frac{\overline{R}}{d_2} \tag{6.6}$$

If we use $\overline{\overline{x}}$ as an estimator of μ and $\overline{R}/d_2$ as an estimator of σ, then the parameters of the $\overline{x}$ chart are

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + \frac{3}{d_2\sqrt{n}}\overline{R} \\ \text{Center line} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - \frac{3}{d_2\sqrt{n}}\overline{R} \end{aligned} \tag{6.7}$$

If we define

$$A_2 = \frac{3}{d_2\sqrt{n}} \tag{6.8}$$

then equation (6.7) reduces to equation (6.4).

Now consider the R chart. The center line will be $\bar{R}$. To determine the control limits, we need an estimate of σ_R. Assuming that the quality characteristic is normally distributed, $\hat{\sigma}_R$ can be found from the distribution of the relative range $W = R/\sigma$. The standard deviation of W, say d_3, is a known function of n. Thus, since

$$R = W\sigma$$

the standard deviation of R is

$$\sigma_R = d_3\sigma$$

Since σ is unknown, we may estimate σ_R by

$$\hat{\sigma}_R = d_3 \frac{\bar{R}}{d_2} \tag{6.9}$$

Consequently, the parameters of the R chart with the usual three-sigma control limits are

$$\begin{aligned} \text{UCL} &= \bar{R} + 3\hat{\sigma}_R = \bar{R} + 3d_3 \frac{\bar{R}}{d_2} \\ \text{Center line} &= \bar{R} \\ \text{LCL} &= \bar{R} - 3\hat{\sigma}_R = \bar{R} - 3d_3 \frac{\bar{R}}{d_2} \end{aligned} \tag{6.10}$$

If we let

$$D_3 = 1 - 3\frac{d_3}{d_2} \quad \text{and} \quad D_4 = 1 + 3\frac{d_3}{d_2}$$

equation (6.10) reduces to equation (6.5).

Control Limits for the $\bar{x}$ Chart

$$\text{UCL} = \bar{\bar{x}} + A_2\bar{R}$$

$$\text{Center line} = \bar{\bar{x}} \qquad (6.4)$$

$$\text{LCL} = \bar{\bar{x}} - A_2\bar{R}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Limits for the R Chart

$$\text{UCL} = D_4\bar{R}$$

$$\text{Center line} = \bar{R} \qquad (6.5)$$

$$\text{LCL} = D_3\bar{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

Phase I Application of $\bar{x}$ and *R* Charts

- Eqns 6.4 and 6.5 are **trial control limits**
  - Determined from *m* initial samples
    - Typically 20-25 subgroups of size *n* between 3 and 5
  - Any out-of-control points should be examined for assignable causes
    - If assignable causes are found, discard points from calculations and revise the trial control limits
    - Continue examination until all points plot in control
    - Adopt resulting trial control limits for use
    - If no assignable cause is found, there are two options
      1. Eliminate point as if an assignable cause were found and revise limits
      2. Retain point and consider limits appropriate for control
  - If there are many out-of-control points they should be examined for **patterns** that may identify underlying process problems

Example 6.1 The Hard Bake Process

■ **TABLE 6.1**

Flow Width Measurements (microns) for the Hard-Bake Process

Sample Number	Wafers 1	2	3	4	5	$\bar{x}_i$	R_i
1	1.3235	1.4128	1.6744	1.4573	1.6914	1.5119	0.3679
2	1.4314	1.3592	1.6075	1.4666	1.6109	1.4951	0.2517
3	1.4284	1.4871	1.4932	1.4324	1.5674	1.4817	0.1390
4	1.5028	1.6352	1.3841	1.2831	1.5507	1.4712	0.3521
5	1.5604	1.2735	1.5265	1.4363	1.6441	1.4882	0.3706
6	1.5955	1.5451	1.3574	1.3281	1.4198	1.4492	0.2674
7	1.6274	1.5064	1.8366	1.4177	1.5144	1.5805	0.4189
8	1.4190	1.4303	1.6637	1.6067	1.5519	1.5343	0.2447
9	1.3884	1.7277	1.5355	1.5176	1.3688	1.5076	0.3589
10	1.4039	1.6697	1.5089	1.4627	1.5220	1.5134	0.2658
11	1.4158	1.7667	1.4278	1.5928	1.4181	1.5242	0.3509
12	1.5821	1.3355	1.5777	1.3908	1.7559	1.5284	0.4204
13	1.2856	1.4106	1.4447	1.6398	1.1928	1.3947	0.4470
14	1.4951	1.4036	1.5893	1.6458	1.4969	1.5261	0.2422
15	1.3589	1.2863	1.5996	1.2497	1.5471	1.4083	0.3499
16	1.5747	1.5301	1.5171	1.1839	1.8662	1.5344	0.6823
17	1.3680	1.7269	1.3957	1.5014	1.4449	1.4874	0.3589
18	1.4163	1.3864	1.3057	1.6210	1.5573	1.4573	0.3153
19	1.5796	1.4185	1.6541	1.5116	1.7247	1.5777	0.3062
20	1.7106	1.4412	1.2361	1.3820	1.7601	1.5060	0.5240
21	1.4371	1.5051	1.3485	1.5670	1.4880	1.4691	0.2185
22	1.4738	1.5936	1.6583	1.4973	1.4720	1.5390	0.1863
23	1.5917	1.4333	1.5551	1.5295	1.6866	1.5592	0.2533
24	1.6399	1.5243	1.5705	1.5563	1.5530	1.5688	0.1156
25	1.5797	1.3663	1.6240	1.3732	1.6887	1.5264	0.3224
						$\Sigma\bar{x}_i$ = 37.6400	ΣR_i = 8.1302
						$\bar{\bar{x}}$ = 1.5056	$\bar{R}$ = 0.32521

$$\overline{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

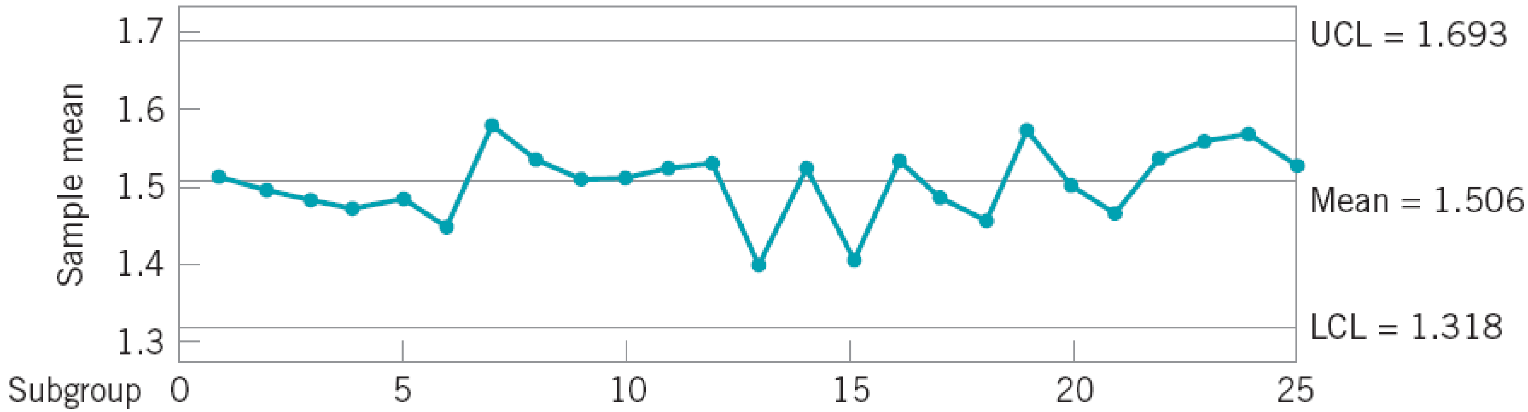
$$\text{LCL} = \overline{R} D_3 = 0.32521(0) = 0$$
$$\text{UCL} = \overline{R} D_4 = 0.32521(2.114) = 0.68749$$

$$\overline{\overline{x}} = \frac{\sum_{i=1}^{25} \overline{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$

$$\text{UCL} = \overline{\overline{x}} + A_2 \overline{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$

and

$$\text{LCL} = \overline{\overline{x}} - A_2 \overline{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$



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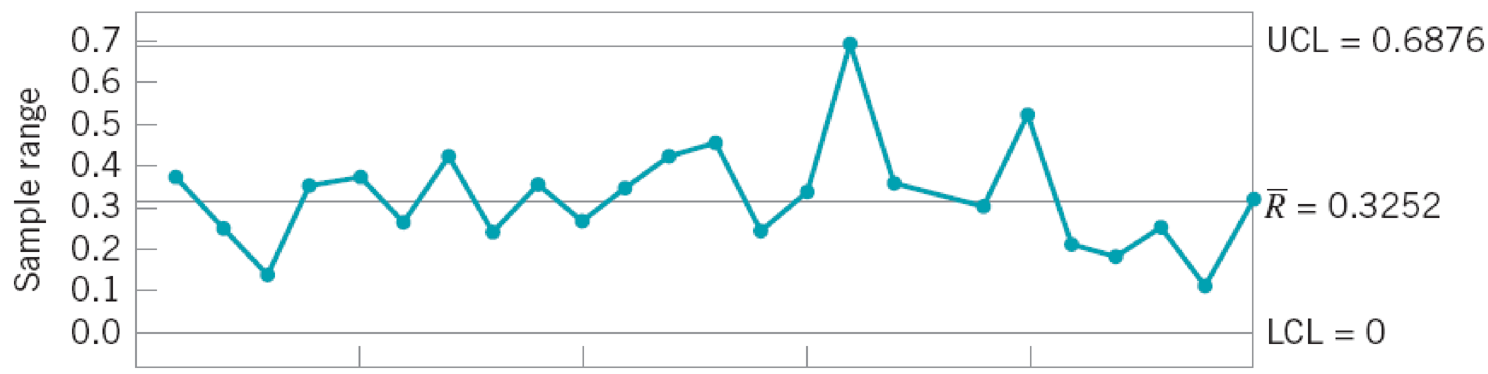


(*b*)

■ **FIGURE 6.2** $\overline{x}$ and R charts (from Minitab) for flow width in the hard-bake process.

Estimating Process Capability

The $\bar{x}$ and R charts provide information about the performance or **capability** of the process. From the $\bar{x}$ chart, we may estimate the mean flow width of the resist in the hard-bake process as $\bar{\bar{x}} = 1.5056$ microns. The process standard deviation may be estimated using equation 5-6; that is,

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{0.32521}{2.326} = 0.1398 \text{ microns}$$

where the value of d_2 for samples of size five is found in Appendix Table VI. The specification limits on flow width are 1.50 ± 0.50 microns. The control chart data may be used to describe the capability of the process to produce wafers relative to these specifications. Assuming that flow width is a normally distributed random variable, with mean 1.5056 and standard deviation 0.1398, we may estimate the fraction of nonconforming wafers produced as

$$\begin{aligned} p &= P\{x < 1.00\} + P\{x > 2.00\} \\ &= \Phi\left(\frac{1.00 - 1.5056}{0.1398}\right) + 1 - \Phi\left(\frac{2.00 - 1.5056}{0.1398}\right) \\ &= \Phi(-3.61660) + 1 - \Phi(3.53648) \\ &\simeq 0.00015 + 1 - 0.99980 \\ &\simeq 0.00035 \end{aligned}$$

That is, about 0.035 percent [350 parts per million (ppm)] of the wafers produced will be outside of the specifications.

Another way to express process capability is in terms of the **process capability ratio (PCR) C_p,** which for a quality characteristic with both upper and lower specification limits (USL and LSL, respectively) is

$$C_p = \frac{\text{USL} - \text{LSL}}{6\sigma} \tag{6.11}$$

Note that the 6σ spread of the process is the basic definition of process capability. Since σ is usually unknown, we must replace it with an estimate. We frequently use $\hat{\sigma} = \overline{R}/d_2$ as an estimate of σ, resulting in an estimate $\hat{C}_p$ of C_p. For the hard-bake process, since $\overline{R}/d_2 = \hat{\sigma} = 0.1398$, we find that

$$\hat{C}_p = \frac{2.00 - 1.00}{6(0.1398)} = \frac{1.00}{0.8388} = 1.192$$

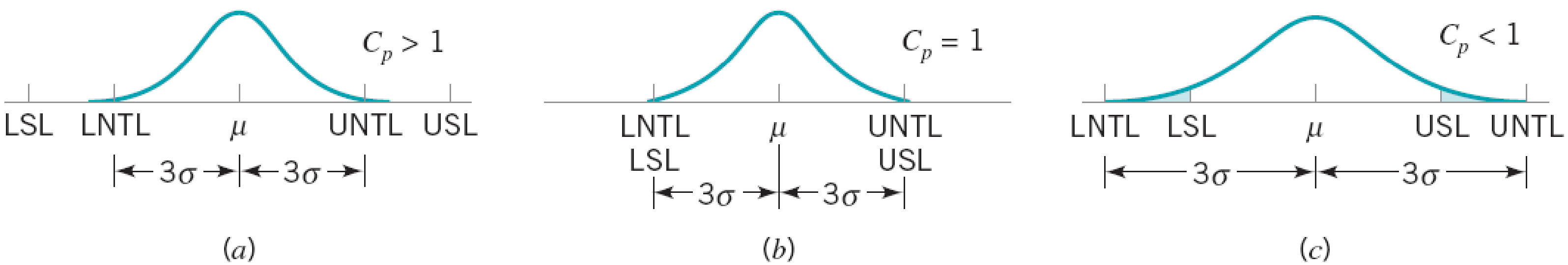
This implies that the "natural" tolerance limits in the process (three-sigma above and below the mean) are inside the lower and upper specification limits. Consequently, a moderately small number of nonconforming wafers will be produced. The PCR C_p may be interpreted another way. The quantity

$$P = \left(\frac{1}{C_p}\right)100\%$$

is simply the percentage of the specification band that the process uses up. For the hard-bake process an estimate of P is

$$\hat{P} = \left(\frac{1}{\hat{C}_p}\right)100\% = \left(\frac{1}{1.192}\right)100\% = 83.89$$

That is, the process uses up about 84% of the specification band.



■ **FIGURE 6.3** Process fallout and the process capability ratio C_p.

Revision of Control Limits and Center Lines

- Effective use of control charts requires periodic review and revision of control limits and center lines
- Sometimes users replace the center line on the $\overline{x}$ chart with a target value
- When R chart is out of control, out-of-control points are often eliminated to recompute a revised value of $\overline{R}$ which is used to determine new limits and center line on R chart and new limits on $\overline{x}$ chart

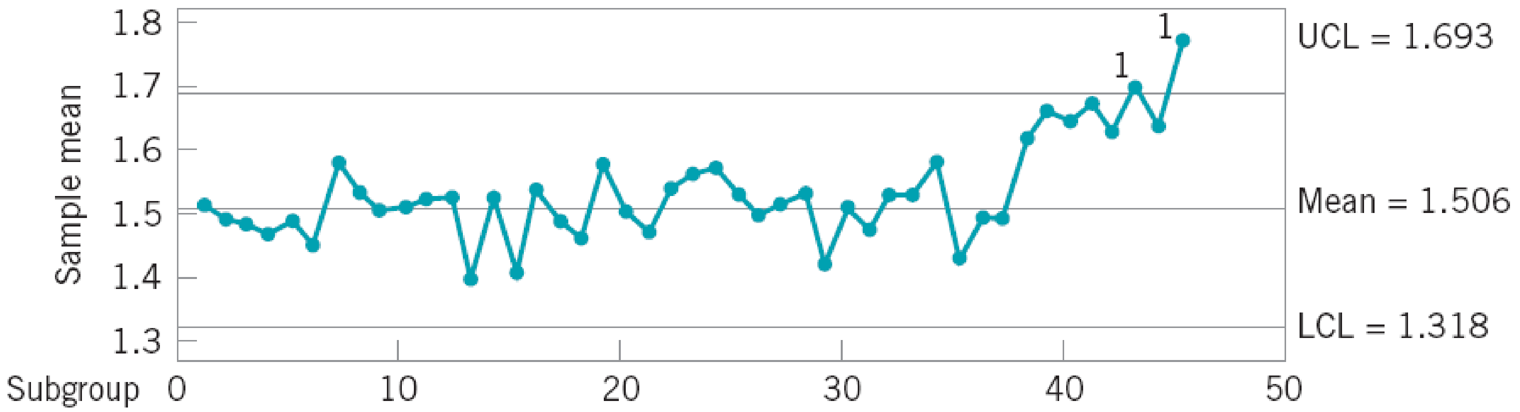
Phase II Operation of Charts

- Use of control chart for monitoring future production, once a set of reliable limits are established, is called *phase II* of control chart usage (Figure 6.4)
- A run chart showing individuals observations in each sample, called a **tolerance chart** or **tier diagram** (Figure 6.5), may reveal patterns or unusual observations in the data

■ TABLE 6.2

Additional Samples for Example 6.1

Sample Number	Wafers						
	1	2	3	4	5	$\bar{x}_i$	R_i
26	1.4483	1.5458	1.4538	1.4303	1.6206	1.4998	0.1903
27	1.5435	1.6899	1.5830	1.3358	1.4187	1.5142	0.3541
28	1.5175	1.3446	1.4723	1.6657	1.6661	1.5332	0.3215
29	1.5454	1.0931	1.4072	1.5039	1.5264	1.4152	0.4523
30	1.4418	1.5059	1.5124	1.4620	1.6263	1.5097	0.1845
31	1.4301	1.2725	1.5945	1.5397	1.5252	1.4724	0.3220
32	1.4981	1.4506	1.6174	1.5837	1.4962	1.5292	0.1668
33	1.3009	1.5060	1.6231	1.5831	1.6454	1.5317	0.3445
34	1.4132	1.4603	1.5808	1.7111	1.7313	1.5793	0.3181
35	1.3817	1.3135	1.4953	1.4894	1.4596	1.4279	0.1818
36	1.5765	1.7014	1.4026	1.2773	1.4541	1.4824	0.4241
37	1.4936	1.4373	1.5139	1.4808	1.5293	1.4910	0.0920
38	1.5729	1.6738	1.5048	1.5651	1.7473	1.6128	0.2425
39	1.8089	1.5513	1.8250	1.4389	1.6558	1.6560	0.3861
40	1.6236	1.5393	1.6738	1.8698	1.5036	1.6420	0.3662
41	1.4120	1.7931	1.7345	1.6391	1.7791	1.6716	0.3811
42	1.7372	1.5663	1.4910	1.7809	1.5504	1.6252	0.2899
43	1.5971	1.7394	1.6832	1.6677	1.7974	1.6970	0.2003
44	1.4295	1.6536	1.9134	1.7272	1.4370	1.6321	0.4839
45	1.6217	1.8220	1.7915	1.6744	1.9404	1.7700	0.3187



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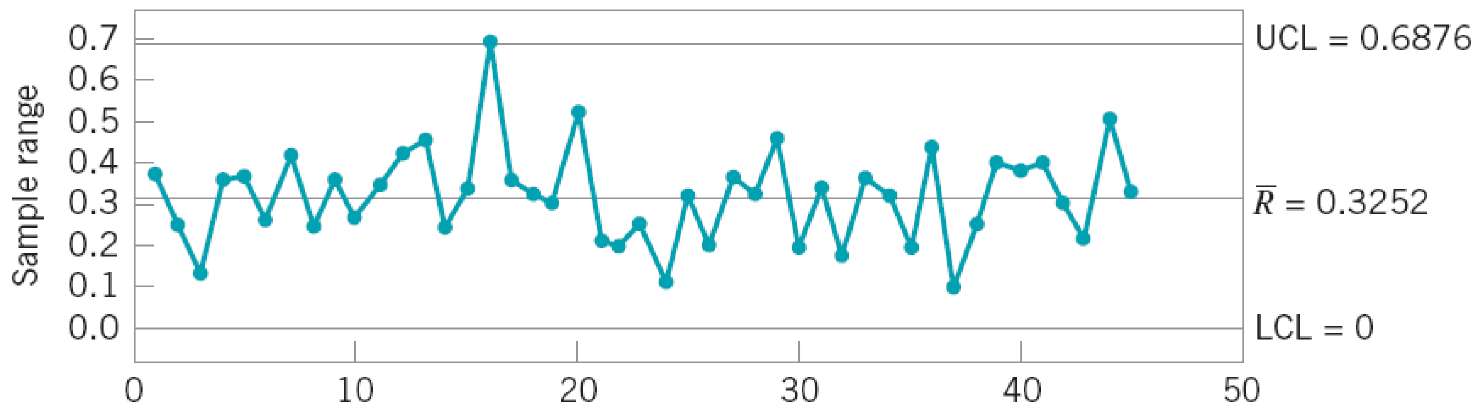
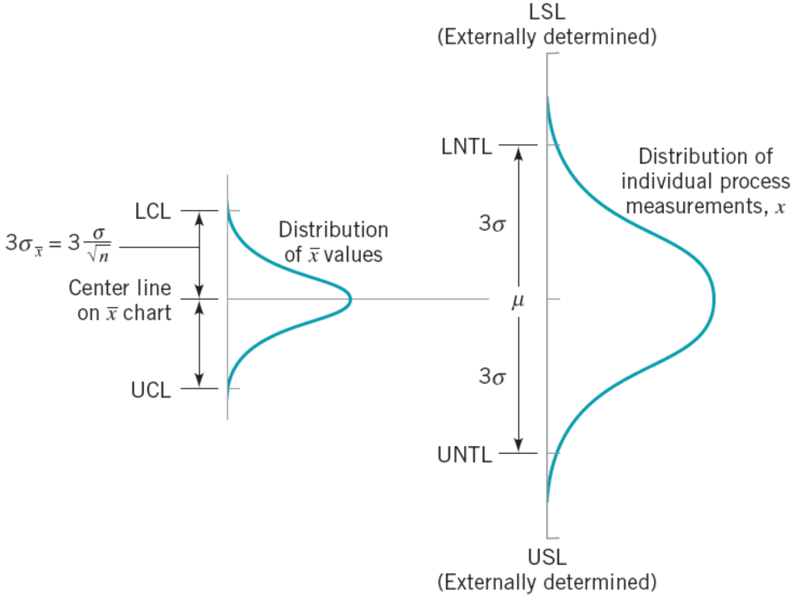


(*b*)

■ **FIGURE 6.4** Continuation of the $\overline{x}$ and R charts in Example 6.1.

Control vs. Specification Limits

- **Control** limits are derived from natural process variability, or the n**atural tolerance** limits of a process
- **Specification** limits are determined externally, for example by customers or designers
- There is no mathematical or statistical relationship between the control limits and the specification limits



■ FIGURE 6.6 Relationship of natural tolerance limits, control limits, and specification limits.

Rational Subgroups

- $\bar{x}$ charts monitor **between-sample variability**
- R charts measure **within-sample variability**
- Standard deviation estimate of σ used to construct control limits is calculated from **within-sample variability**
- It is not correct to estimate σ using

$$s = \sqrt{\frac{\sum_{i=1}^{m}\sum_{j=1}^{n}\left(x_{ij} - \bar{\bar{x}}\right)^2}{mn-1}}$$

Guidelines for Control Chart Design

- Control chart design requires specification of sample size, control limit width, and sampling frequency
  - Exact solution requires detailed information on statistical characteristics as well as economic factors
  - The problem of choosing sample size and sampling frequency is one of **allocating sampling effort**
- For $\overline{X}$ chart, choose as small a sample size is consistent with magnitude of process shift one is trying to detect. For moderate to large shifts, relatively small samples are effective. For small shifts, larger samples are needed.
- For small samples, R chart is relatively insensitive to changes in process standard deviation. For larger samples ($n > 10$ or 12), s or s^2 charts are better choices.

Probability Limits on the $\bar{x}$ and R Charts

It is customary to express the control limits on the $\bar{x}$ and R charts as a multiple of the standard deviation of the statistic plotted on the charts. If the multiple chosen is k, then the limits are referred to as k-sigma limits, the usual choice being $k = 3$. As mentioned in Chapter 4, however, it is also possible to define the control limits by specifying the type I error level for the test. Such control limits are called probability limits and are used extensively in the United Kingdom and some parts of Western Europe.

It is easy to choose probability limits for the $\bar{x}$ chart. Since $\bar{x}$ is approximately normally distributed, we may obtain a desired type I error of α by choosing the multiple of sigma for the control limit as $k = Z_{\alpha/2}$, where $Z_{\alpha/2}$ is the upper $\alpha/2$ percentage point of the standard normal distribution. Note that the usual three-sigma limits imply that the type I error probability is $\alpha = 0.0027$. If we choose $\alpha = 0.002$, for example, as most writers who recommend probability limits do, then $Z_{\alpha/2} = Z_{0.001} = 3.09$. Consequently, there is very little difference between such control limits and three-sigma control limits.

We may also construct R charts using probability limits. If $\alpha = 0.002$, the 0.001 and 0.999 percentage points of the distribution of the relative range $W = R/\sigma$ are required. These points obviously depend on the subgroup size n. Denoting these points by $W_{0.001}(n)$ and $W_{0.999}(n)$, and estimating σ by $\bar{R}/d_2$, we would have the 0.001 and 0.999 limits for R as $W_{0.001}(n)(\bar{R}/d_2)$ and $W_{0.999}(n)(\bar{R}/d_2)$. If we let $D_{0.001} = W_{0.001}(n)/d_2$ and $D_{0.999} = W_{0.999}(n)/d_2$, then the probability limits for the R chart are

$$\text{UCL} = D_{0.999}\bar{R}$$

$$\text{LCL} = D_{0.001}\bar{R}$$

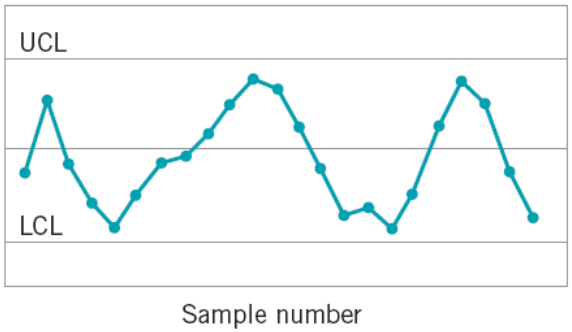
6.2.3 Charts Based on Standard Values

$$
\begin{aligned}
\text{UCL} &= \mu + 3\frac{\sigma}{\sqrt{n}} \\
\text{Center line} &= \mu \\
\text{LCL} &= \mu - 3\frac{\sigma}{\sqrt{n}}
\end{aligned}
\qquad (6.14)
$$

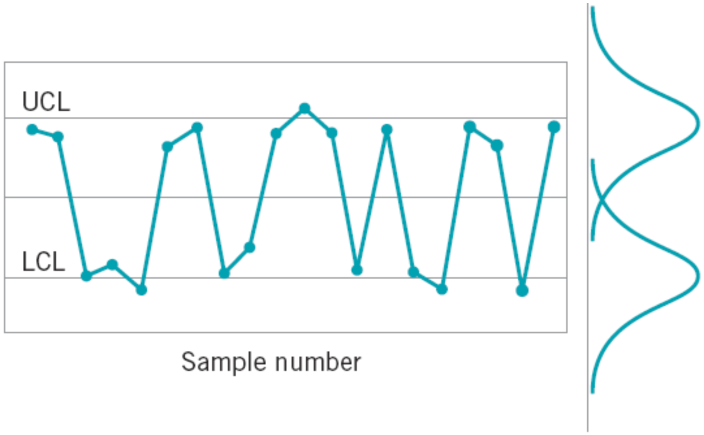
$$
\begin{aligned}
\text{UCL} &= \mu + A\sigma \\
\text{Center line} &= \mu \\
\text{LCL} &= \mu - A\sigma
\end{aligned}
\qquad (6.15)
$$

$$
\begin{aligned}
\text{UCL} &= D_2\sigma \\
\text{Center line} &= d_2\sigma \\
\text{LCL} &= D_1\sigma
\end{aligned}
\qquad (6.17)
$$

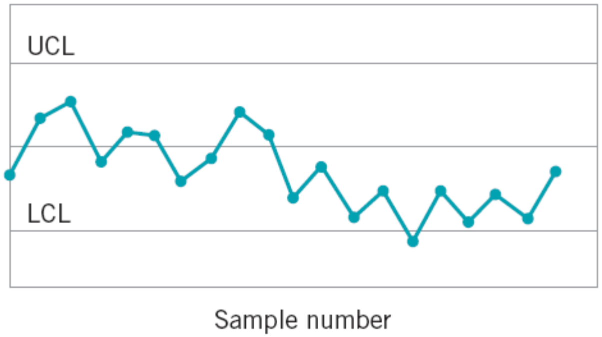
6.2.4 Interpretation of $\overline{x}$ and R Charts



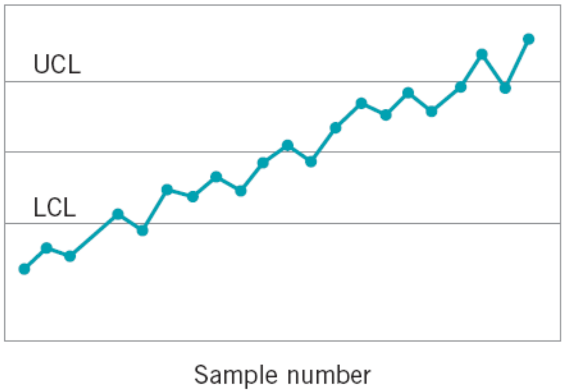
■ **FIGURE 6.8** Cycles on a control chart.



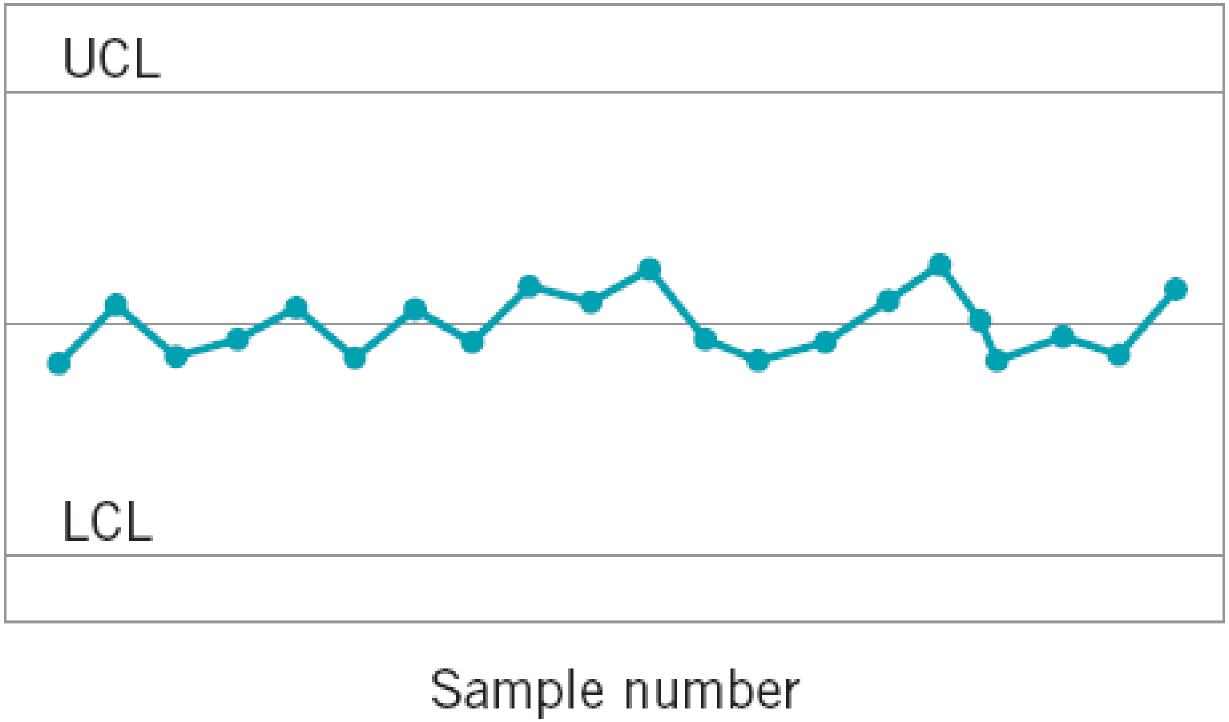
■ **FIGURE 6.9** A mixture pattern.



■ **FIGURE 6.10** A shift in process level.



■ **FIGURE 6.11** A trend in process level.



■ FIGURE 6.12 A stratification pattern.

6.2.5 The Effect of Non-normality on $\bar{x}$ and R Charts

- An assumption in performance properties is that the underlying distribution of quality characteristic is **normal**
  - If underlying distribution is not normal, sampling distributions can be derived and exact probability limits obtained
- Burr (1967) notes the usual normal theory control limits are very robust to normality assumption
- Schilling and Nelson (1976) indicate that in most cases, samples of size 4 or 5 are sufficient to ensure reasonable robustness to normality assumption for $\bar{x}$ chart
- Sampling distribution of R is **not** symmetric, thus symmetric 3-sigma limits are an approximation and α-risk is not 0.0027. R chart is more sensitive to departures from normality than $\bar{x}$ chart.
- Assumptions of normality and independence are not a primary concern in phase I

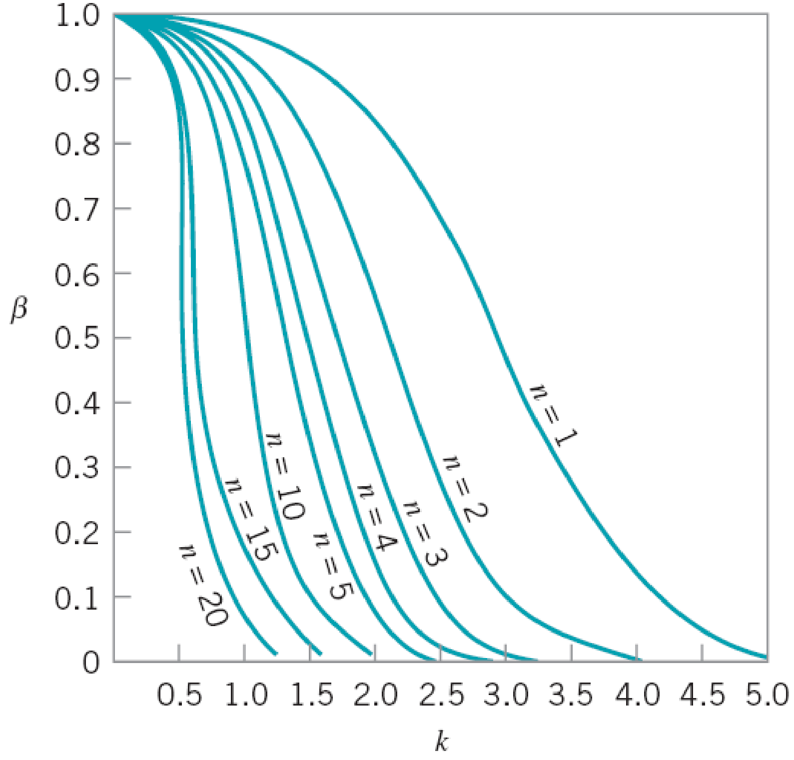
6.2.6 The Operating-Characteristic Function

$$\beta = \Phi\left(L - k\sqrt{n}\right) - \Phi\left(-L - k\sqrt{n}\right) \tag{6.19}$$

To illustrate the use of equation (6.19), suppose that we are using an $\bar{x}$ chart with $L = 3$ (the usual three-sigma limits), the sample size $n = 5$, and we wish to determine the probability of detecting a shift to $\mu_1 = \mu_0 + 2\sigma$ on the first sample following the shift. Then, since $L = 3$, $k = 2$, and $n = 5$, we have

$$\begin{aligned}\beta &= \Phi\left[3 - 2\sqrt{5}\right] - \Phi\left[-3 - 2\sqrt{5}\right]\\ &= \Phi(-1.47) - \Phi(-7.37)\\ &\cong 0.0708\end{aligned}$$

This is the β-risk, or the probability of not detecting such a shift. The probability that such a shift *will* be detected on the first subsequent sample is $1 - \beta = 1 - 0.0708 = 0.9292$.



If the shift is 1.0σ and the sample size is n = 5, then β = 0.75.

■ **FIGURE 6.13** Operating-characteristic curves for the $\overline{x}$ chart with three-sigma limits. $\beta = P$ (not detecting a shift of $k\sigma$ in the mean on the first sample following the shift).

In general, the expected number of samples taken before the shift is detected is simply the **average run length,** or

$$\text{ARL} = \sum_{r=1}^{\infty} r\beta^{r-1}(1-\beta) = \frac{1}{1-\beta}$$

Therefore, in our example, we have

$$\text{ARL} = \frac{1}{1-\beta} = \frac{1}{0.25} = 4$$

In other words, the expected number of samples taken to detect a shift of 1.0σ with $n = 5$ is 4.

6.2.7 The Average Run Length for the $\bar{x}$ Chart

For any **Shewhart control chart,** we have noted previously that the ARL can be expressed as

$$\text{ARL} = \frac{1}{P(\text{one point plots out of control})}$$

or

$$\text{ARL}_0 = \frac{1}{\alpha} \tag{6.20}$$

for the in-control ARL and

$$\text{ARL}_1 = \frac{1}{1-\beta} \tag{6.21}$$

for the out-of-control ARL. These results are actually rather intuitive. If the observations plotted on the control chart are independent, then the number of points that must be plotted until the first point exceeds a control limit is a geometric random variable with parameter p (see Chapter 3). The mean of this geometric distribution is simply $1/p$, the average run length.

Two other performance measures based on ARL are sometimes of interest. The average time to signal is the number of time periods that occur until a signal is generated on the control chart. If samples are taken at equally spaced intervals of time h, then the **average time to signal** or the ATS is

$$\text{ATS} = \text{ARL}\, h \tag{6.22}$$

It may also be useful to express the ARL in terms of the expected number of individual *units* sampled—say, I—rather than the number of samples taken to detect a shift. If the sample size is n, the relationship between I and ARL is

$$I = n\, \text{ARL} \tag{6.23}$$

6.3 Control Charts for $\bar{x}$ and s

Although $\bar{x}$ and R charts are widely used, it is occasionally desirable to estimate the process standard deviation directly instead of indirectly through the use of the range R. This leads to control charts for $\bar{x}$ and s, where s is the sample standard deviation.[1] Generally, $\bar{x}$ and s charts are preferable to their more familiar counterparts, $\bar{x}$ and R charts, when either

1. The sample size n is moderately large—say, $n > 10$ or 12. (Recall that the range method for estimating σ loses statistical efficiency for moderate to large samples.)
2. The sample size n is variable.

In this section, we illustrate the construction and operation of $\bar{x}$ and s control charts. We also show how to deal with variable sample size and discuss an alternative to the s chart.

$$\begin{aligned} \text{UCL} &= B_4\bar{s} \\ \text{Center line} &= \bar{s} \\ \text{LCL} &= B_3\bar{s} \end{aligned} \tag{6.27}$$

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_3\bar{s} \\ \text{Center line} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - A_3\bar{s} \end{aligned} \tag{6.28}$$

■ TABLE 6.3

Inside Diameter Measurements (mm) for Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	74.011	74.004	74.001	0.0075
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985	73.993	73.996	0.0087
7	73.995	74.006	73.994	74.000	74.005	74.000	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005	74.004	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998	73.997	74.012	73.998	0.0105
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998	73.999	74.007	74.006	0.0073
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005	74.007	74.001	0.0106
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013	74.020	74.003	74.009	0.0080
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162
						$\Sigma =$ 1850.028	0.2351
						$\bar{\bar{x}} =$ 74.001	$\bar{s} =$ 0.0094

Development of the control limits:

If no standard is given for σ, then it must be estimated by analyzing past data. Suppose that m preliminary samples are available, each of size n, and let s_i be the standard deviation of the ith sample. The average of the m standard deviations is

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$$

The statistic $\bar{s}/c_4$ is an unbiased estimator of σ. Therefore, the parameters of the s chart would be

$$\text{UCL} = \bar{s} + 3\frac{\bar{s}}{c_4}\sqrt{1-c_4^2}$$

$$\text{Center line} = \bar{s}$$

$$\text{LCL} = \bar{s} - 3\frac{\bar{s}}{c_4}\sqrt{1-c_4^2}$$

We usually define the constants

$$B_3 = 1 - \frac{3}{c_4}\sqrt{1-c_4^2} \quad \text{and} \quad B_4 = 1 + \frac{3}{c_4}\sqrt{1-c_4^2} \qquad (6.26)$$

When $\bar{s}/c_4$ is used to estimate σ, we may define the control limits on the corresponding $\bar{x}$ chart as

$$\text{UCL} = \bar{\bar{x}} + \frac{3\bar{s}}{c_4\sqrt{n}}$$

$$\text{Center line} = \bar{\bar{x}}$$

$$\text{LCL} = \bar{\bar{x}} - \frac{3\bar{s}}{c_4\sqrt{n}}$$

Let the constant $A_3 = 3/(c_4\sqrt{n})$.

EXAMPLE 6.3 $\bar{x}$ and s Charts for the Piston Ring Data

Construct and interpret $\bar{x}$ and s charts using the piston ring inside diameter measurements in Table 6.3.

SOLUTION

The grand average and the average standard deviation are

$$\bar{\bar{x}} = \frac{1}{25}\sum_{i=1}^{25} \bar{x}_i = \frac{1}{25}(1850.028) = 74.001$$

and

$$\bar{s} = \frac{1}{25}\sum_{i=1}^{25} s_i = \frac{1}{25}(0.2351) = 0.0094$$

respectively. Consequently, the parameters for the $\bar{x}$ chart are

$$\text{UCL} = \bar{\bar{x}} + A_3\bar{s} = 74.001 + (1.427)(0.0094) = 74.014$$

$$\text{CL} = \bar{\bar{x}} = 74.001$$

$$\text{LCL} = \bar{\bar{x}} - A_3\bar{s} = 74.001 - (1.427)(0.0094) = 73.988$$

and for the s chart

$$\text{UCL} = B_4\bar{s} = (2.089)(0.0094) = 0.0196$$

$$\text{CL} = \bar{s} = 0.0094$$

$$\text{LCL} = B_3\bar{s} = (0)(0.0094) = 0$$

The control charts are shown in Fig. 6.17. There is no indication that the process is out of control, so those limits could be adopted for phase II monitoring of the process.

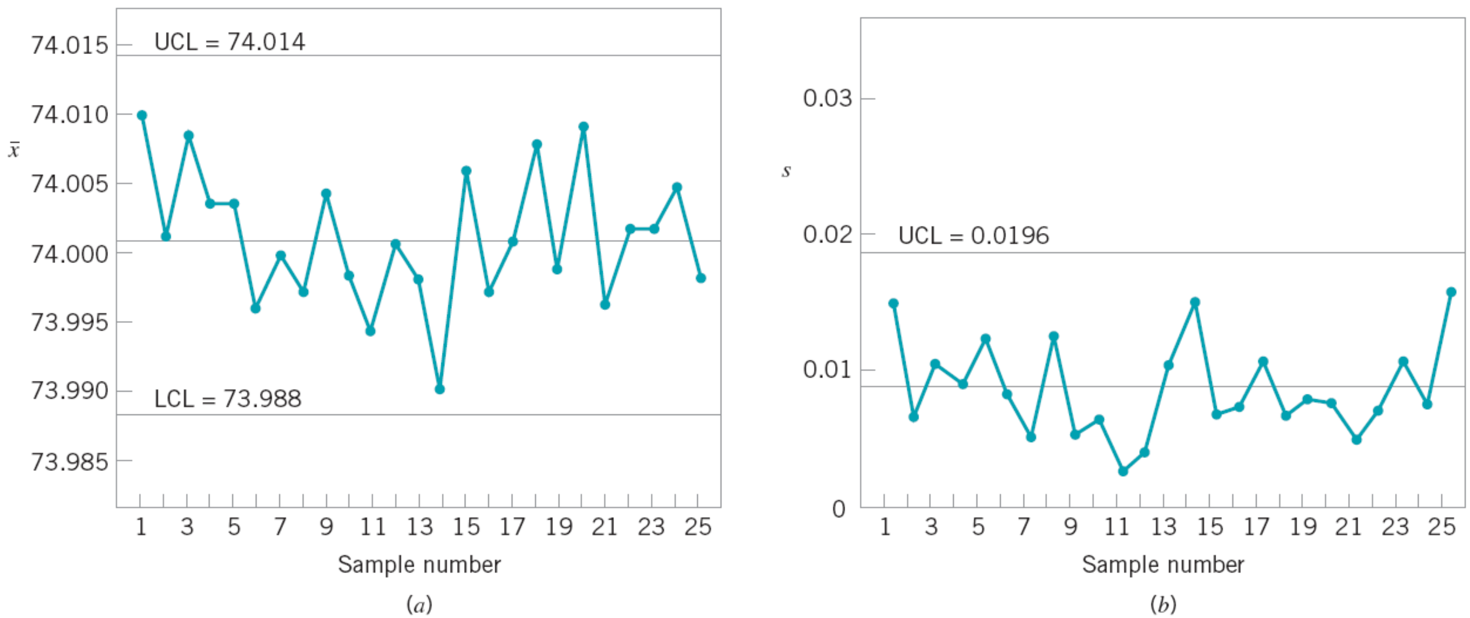


FIGURE 6.17 The $\bar{x}$ and s control charts for Example 6.3. (*a*) The $\bar{x}$ chart with control limits based on $\bar{s}$. (*b*) The s control chart.

6.3.2 The $\overline{x}$ and s Control Charts with Variable Sample Size

The $\overline{x}$ and s control charts are relatively easy to apply in cases where the sample sizes are variable. In this case, we should use a weighted average approach in calculating $\overline{\overline{x}}$ and $\overline{s}$. If n_i is the number of observations in the ith sample, then use

$$\overline{\overline{x}} = \frac{\sum_{i=1}^{m} n_i \overline{x}_i}{\sum_{i=1}^{m} n_i} \tag{6.30}$$

and

$$\overline{s} = \left[\frac{\sum_{i=1}^{m} (n_i - 1) s_i^2}{\sum_{i=1}^{m} n_i - m} \right]^{1/2} \tag{6.31}$$

as the center lines on the $\overline{x}$ and s control charts, respectively. The control limits would be calculated from equations (6.27) and (6.28), respectively, but the constants A_3, B_3, and B_4 will depend on the sample size used in each individual subgroup.

EXAMPLE 6.4 $\bar{x}$ and s Chart for the Piston Rings, Variable Sample Size

Consider the data in Table 6.4, which is a modification of the piston-ring data used in Example 6.3. Note that the sample sizes vary from $n = 3$ to $n = 5$. Use the procedure described on page 255 to set up the $\bar{x}$ and s control charts.

SOLUTION

The weighted grand mean and weighted average standard deviation are computed from equations (6.30) and (6.31) as follows:

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} n_i \bar{x}_i}{\sum_{i=1}^{25} n_i} = \frac{5(74.010) + 3(73.996) + \cdots + 5(73.998)}{5 + 3 + \cdots + 5}$$

$$= \frac{8362.075}{113} = 74.001$$

■ **TABLE 6.4**

Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001			73.996	0.0046
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985		73.996	0.0099
7	73.995	74.006	73.994	74.000		73.999	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005		74.004	0.0064
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998			73.994	0.0100
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998			74.008	0.0087
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005		73.999	0.0115
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013			74.008	0.0068
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

and

$$\bar{s} = \left[\frac{\sum_{i=1}^{25}(n_i - 1)s_i^2}{\sum_{i=1}^{25} n_i - 25}\right]^{1/2} = \left[\frac{4(0.0148)^2 + 2(0.0046)^2 + \cdots + 4(0.0162)^2}{5 + 3 + \cdots + 5 - 25}\right]^{1/2}$$

$$= \left[\frac{0.009324}{88}\right]^{1/2} = 0.0103$$

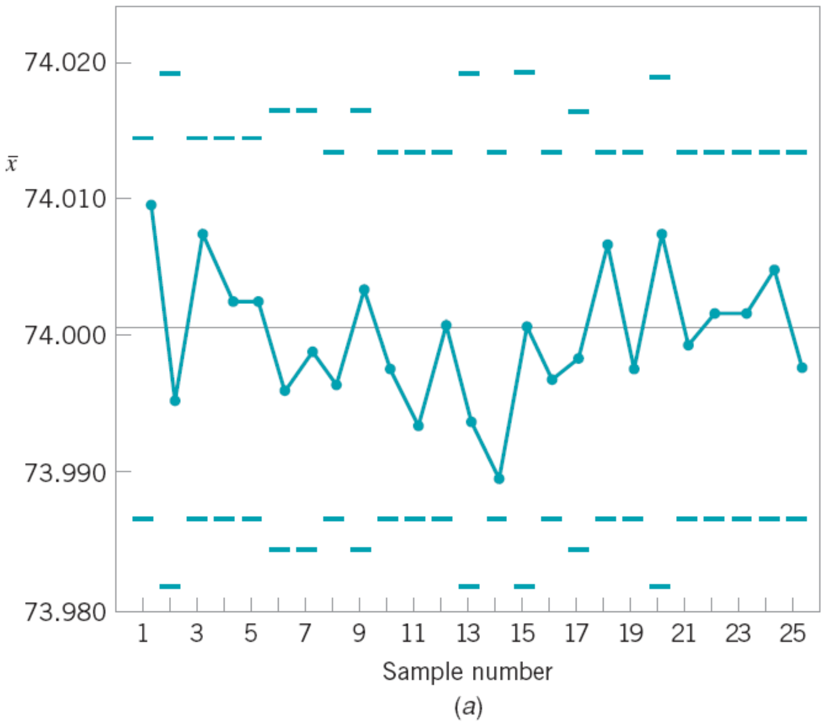
Therefore, the center line of the $\bar{x}$ chart is $\bar{\bar{x}} = 74.001$, and the center line of the s chart is $\bar{s} = 0.0103$. The control limits may now be easily calculated. To illustrate, consider the first sample. The limits for the $\bar{x}$ chart are

$$\text{UCL} = 74.001 \quad 1.427 \quad 0.0103 \; = 74.016$$
$$\text{CL} = 74.001$$
$$\text{LCL} = 74.001 - \; 1.427 \quad 0.0103 \; = 73.986$$

The control limits for the s chart are

$$\text{UCL} = \; 2.089 \quad 0.0103 \; = 0.022$$
$$\text{CL} = 0.0103$$
$$\text{LCL} = 0 \; 0.0103 \; = 0$$

Note that we have used the values of A_3, B_3, and B_4 for $n_1 = 5$. The limits for the second sample would use the values of these constants for $n_2 = 3$. The control limit calculations for all 25 samples are summarized in Table 6.5. The control charts are plotted in Fig. 6.18.



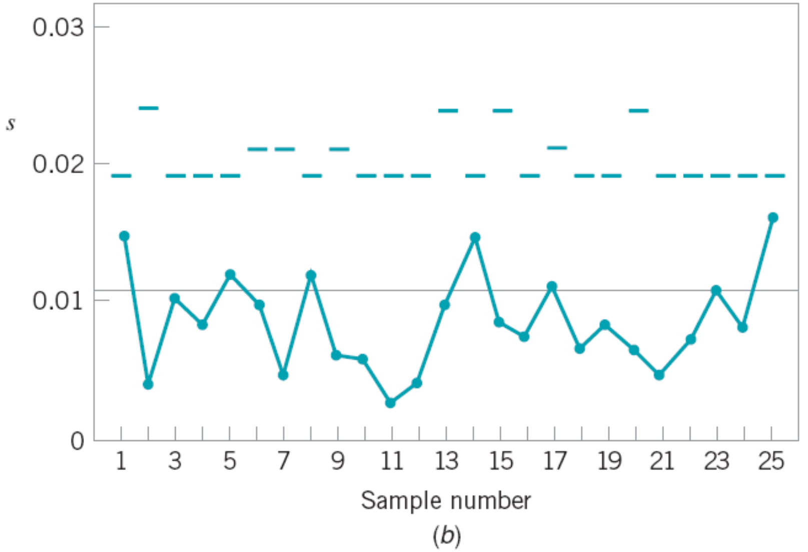


FIGURE 6.18 The (*a*) $\bar{x}$ and (*b*) s control charts for piston-ring data with variable sample size, Example 6.4.

■ TABLE 6.5

Computation of Control Limits for $\bar{x}$ and s Charts with Variable Sample Size

					$\bar{x}$ Chart				s Chart	
Sample	n	$\bar{x}$	s	A_3	LCL	UCL	B_3	B_4	LCL	UCL
1	5	74.010	0.0148	1.427	73.986	74.016	0	2.089	0	0.022
2	3	73.996	0.0046	1.954	73.981	74.021	0	2.568	0	0.026
3	5	74.008	0.0147	1.427	73.986	74.016	0	2.089	0	0.022
4	5	74.003	0.0091	1.427	73.986	74.016	0	2.089	0	0.022
5	5	74.003	0.0122	1.427	73.986	74.016	0	2.089	0	0.022
6	4	73.996	0.0099	1.628	73.984	74.018	0	2.266	0	0.023
7	4	73.999	0.0055	1.628	73.984	74.018	0	2.266	0	0.023
8	5	73.997	0.0123	1.427	73.986	74.016	0	2.089	0	0.022
9	4	74.004	0.0064	1.628	73.984	74.018	0	2.266	0	0.023
10	5	73.998	0.0063	1.427	73.986	74.016	0	2.089	0	0.022
11	5	73.994	0.0029	1.427	73.986	74.016	0	2.089	0	0.022
12	5	74.001	0.0042	1.427	73.986	74.016	0	2.089	0	0.022
13	3	73.994	0.0100	1.954	73.981	74.021	0	2.568	0	0.026
14	5	73.990	0.0153	1.427	73.986	74.016	0	2.089	0	0.022
15	3	74.008	0.0087	1.954	73.981	74.021	0	2.568	0	0.026
16	5	73.997	0.0078	1.427	73.986	74.016	0	2.089	0	0.022
17	4	73.999	0.0115	1.628	73.984	74.018	0	2.226	0	0.023
18	5	74.007	0.0070	1.427	73.986	74.016	0	2.089	0	0.022
19	5	73.998	0.0085	1.427	73.986	74.016	0	2.089	0	0.022
20	3	74.008	0.0068	1.954	73.981	74.021	0	2.568	0	0.026
21	5	74.000	0.0122	1.427	73.986	74.016	0	2.089	0	0.022
22	5	74.002	0.0074	1.427	73.986	74.016	0	2.089	0	0.022
23	5	74.002	0.0119	1.427	73.986	74.016	0	2.089	0	0.022
24	5	74.005	0.0087	1.427	73.986	74.016	0	2.089	0	0.022
25	5	73.998	0.0162	1.427	73.986	74.016	0	2.089	0	0.022

6.3.3 The s^2 Control Chart

Most quality engineers use either the R chart or the s chart to monitor process variability, with s preferable to R for moderate to large sample sizes. Some practitioners recommend a control chart based directly on the sample variance s^2, the **s^2 control chart.** The parameters for the s^2 control chart are

$$
\begin{aligned}
\text{UCL} &= \frac{\bar{s}^2}{n-1}\chi^2_{\alpha/2,n-1} \\
\text{Center line} &= \bar{s}^2 \\
\text{LCL} &= \frac{\bar{s}^2}{n-1}\chi^2_{1-(\alpha/2),n-1}
\end{aligned}
\tag{6.32}
$$

where $\chi^2_{\alpha/2,}$ and $\chi^2_{(\alpha/2),n-1}$ denote the upper and lower $\alpha/2$ percentage points of the chi-square distribution with $n-1$ degrees of freedom, and $\bar{s}^2$ is an average sample variance obtained from the analysis of preliminary data. A standard value σ^2 could be used in equation (6.32) instead of $\bar{s}^2$ if one were available. Note that this control chart is defined with probability limits.

6.4 The Shewhart Control Chart for Individual Measurements

There are many situations in which the sample size used for process monitoring is $n = 1$; that is, the sample consists of an individual unit. Some examples of these situations are as follows:

1. Automated inspection and measurement technology is used, and every unit manufactured is analyzed so there is no basis for rational subgrouping.
2. Data comes available relatively slowly, and it is inconvenient to allow sample sizes of $n > 1$ to accumulate before analysis. The long interval between observations will cause problems with rational subgrouping. This occurs frequently in both manufacturing and nonmanufacturing situations.
3. Repeat measurements on the process differ only because of laboratory or analysis error, as in many chemical processes.
4. Multiple measurements are taken on the same unit of product, such as measuring oxide thickness at several different locations on a wafer in semiconductor manufacturing.
5. In process plants, such as papermaking, measurements on some parameter such as coating thickness *across* the roll will differ very little and produce a standard deviation that is much too small if the objective is to control coating thickness *along* the roll.

In such situations, the **control chart for individual units** is useful. (The cumulative sum and exponentially weighted moving-average control charts discussed in Chapter 9 will be a better alternative in phase II or when the magnitude of the shift in process mean that is of interest is small.) In many applications of the *individuals control chart*, we use the *moving range* two successive observations as the basis of estimating the process variability. The moving range is defined as

$$\mathrm{MR}_i = |x_i - x_{i-1}|$$

It is also possible to establish a **moving range control chart.** The procedure is illustrated in the following example.

EXAMPLE 6.5 Loan Processing Costs

The mortgage loan processing unit of a bank monitors the costs of processing loan applications. The quantity tracked is the average weekly processing costs, obtained by dividing total weekly costs by the number of loans processed during the week. The processing costs for the most recent 20 weeks are shown in Table 6.6. Set up individual and moving range control charts for these data.

■ TABLE 6.6

Costs of Processing Mortgage Loan Applications

Weeks	Cost x	Moving Range MR
1	310	
2	288	22
3	297	9
4	298	1
5	307	9
6	303	4
7	294	9
8	297	3
9	308	11
10	306	2
11	294	12
12	299	5
13	297	2
14	299	2
15	314	15
16	295	19
17	293	2
18	306	13
19	301	5
20	304	3
	$\bar{x} = 300.5$	$\overline{MR} = 7.79$

SOLUTION

To set up the control chart for individual observations, note that the sample average cost of the 20 observations is $\bar{x} = 300.5$ and that the average of the moving ranges of two observations is $\overline{MR} = 7.79$. To set up the moving range chart, we use $D_3 = 0$ and $D_4 = 3.267$ for $n = 2$. Therefore, the moving range chart has center line $\overline{MR} = 7.79$, LCL = 0, and UCL $= D_4\overline{MR} = (3.267)7.79 = 25.45$. The control chart (from Minitab) is shown in Fig. 6.19*b*. Notice that no points are out of control.

For the control chart for individual measurements, the parameters are

$$\begin{aligned} \text{UCL} &= \bar{x} + 3\frac{\overline{MR}}{d_2} \\ \text{Center line} &= \bar{x} \\ \text{LCL} &= \bar{x} - 3\frac{\overline{MR}}{d_2} \end{aligned} \tag{6.33}$$

If a moving range of $n = 2$ observations is used, then $d_2 = 1.128$. For the data in Table 6.6, we have

$$\text{UCL} = \bar{x} + 3\frac{\overline{\text{MR}}}{d_2} = 300.5 + 3\frac{7.79}{1.128} = 321.22$$

$$\text{Center line} = \bar{x} = 34.088$$

$$\text{LCL} = \bar{x} - 3\frac{\overline{\text{MR}}}{d_2} = 300.5 - 3\frac{7.79}{1.128} = 279.78$$

The control chart for individual cost values is shown in Fig. 6.19*a*. There are no out of control observations on the individuals control chart.

The interpretation of the individuals control chart is ve similar to the interpretation of the ordinary $\bar{x}$ control chart. shift in the process mean will result in a single point or a seri of points that plot outside the control limits on the contr chart for individuals. Sometimes a point will plot outside tl control limits on both the individuals chart and the movi range chart. This will often occur because a large value of will also lead to a large value of the moving range for th sample. This is very typical behavior for the individuals aı moving range control charts. It is most likely an indicatio that the mean is out of control and not an indication that bo the mean and the variance of the process are out of control.

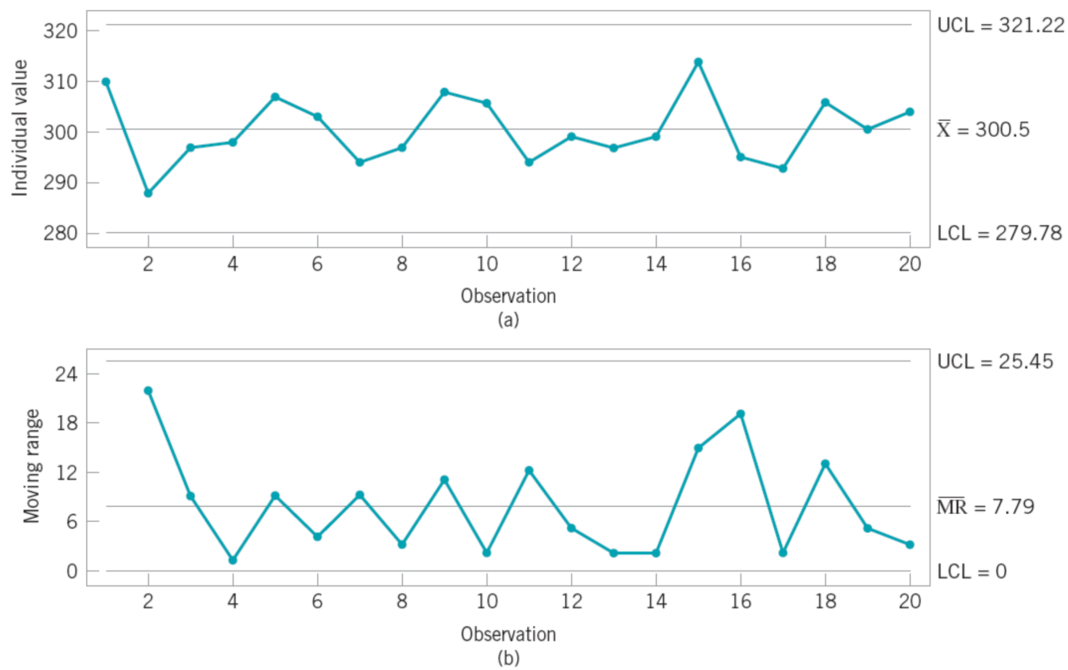


FIGURE 6.19 Control charts for (*a*) individual observations on cost and for (*b*) the moving range.

TABLE 6.7

Costs of Processing Mortgage Loan Applications, Weeks 21–40

Week	Cost x	Week	Cost x
21	305	31	310
22	282	32	292
23	305	33	305
24	296	34	299
25	314	35	304
26	295	36	310
27	287	37	304
28	301	38	305
29	298	39	333
30	311	40	328

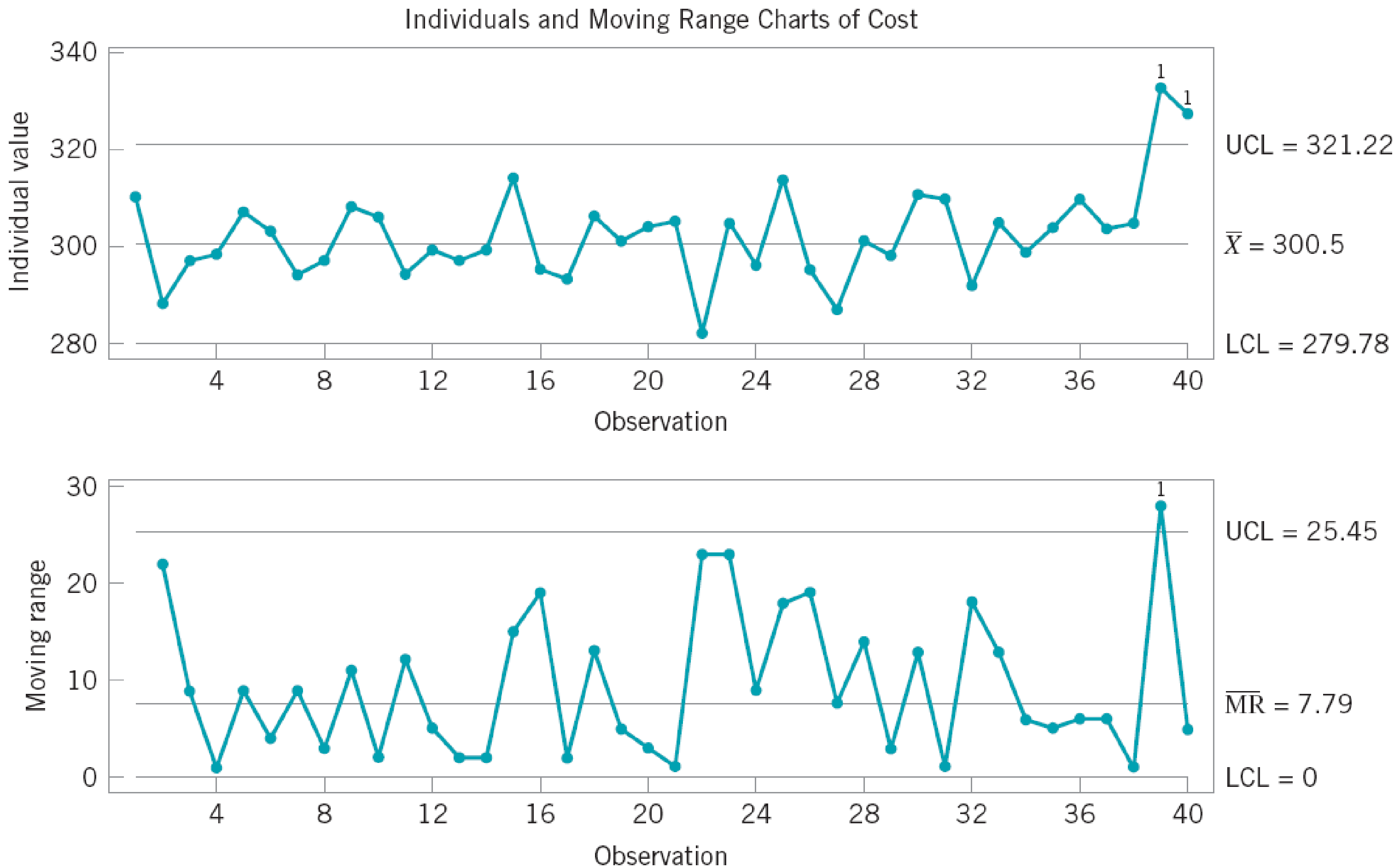


FIGURE 6.20 Continuation of the control chart for individuals and the moving range using the additional data in Table 6.7.

Average Run Lengths

- Crowder (1987b) showed that ARL_0 of combined individuals and moving-range chart with conventional 3-sigma limits is generally much less than ARL_0 (= 370) of standard Shewhart control chart

- Ability of individuals chart to detect small shifts is very poor
  - Rather than narrowing the 3-sigma limits, correct approach to detecting small shifts is a cumulative-sum or exponentially weighted moving-average control chart (Chapter 9)

Size of Shift	β	ARL_1
1σ	0.9772	43.96
2σ	0.8413	6.30
3σ	0.5000	2.00

Normality

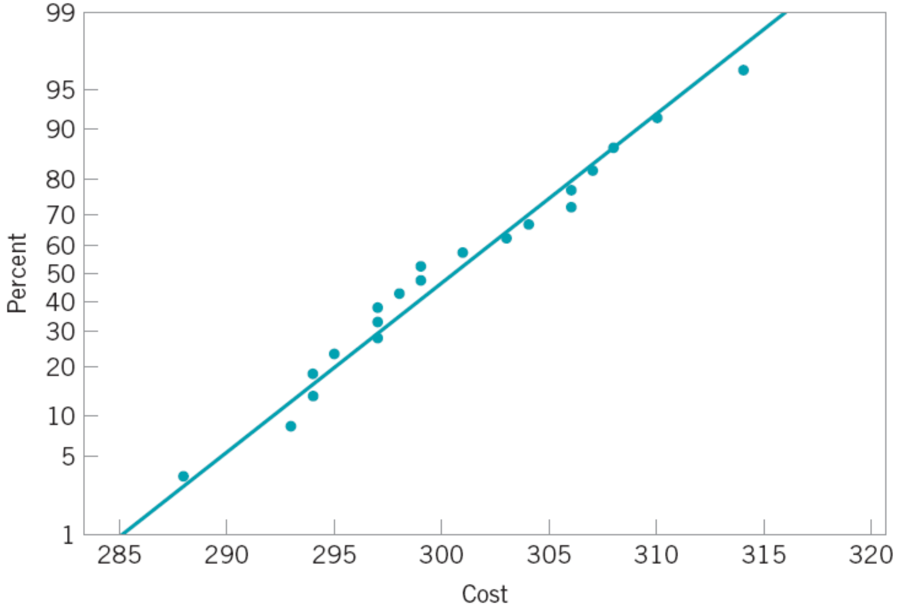


FIGURE 6.21
Normal probability plot of the mortgage application processing cost data from Table 6.6, Example 6.5.

- Borror, Montgomery, and Runger (1999) found in-control ARL is dramatically affected by nonnormal data
- One approach for nonnormal data is to determine control limits for individuals control chart based on percentiles of correct underlying distribution
  - Requires at least 100 and preferably 200 observations