Financial Mathematics

Value at risk and other risk measures

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Value at risk for a given $\alpha \in (0,1)$ is the smallest x such that the probability that the loss L does not exceed x is not larger that $(1-\alpha)$:

$$VaR_\alpha(L) = \inf\{ x \in \mathbb{R} \ : \ P(L > x) \leq 1 - \alpha\} = \inf\{x \in \mathbb{R} \ : \ F_L(x) \geq a\}.$$

Value at risk is a quantile of the loss distribution

Common choices for α are $\alpha = 0.95$ and $\alpha = 0.99$.

Basel III and Solvency II have turned VaR_α into a standard reporting and risk management tool.

For continuous loss distributions $VaR_\alpha(L) = F_L^{-1}(\alpha)$.

Example

Calculate $VaR_\alpha(L)$ if $L \sim N(\mu, \sigma^2)$.

We have that

$$F_L(x) = P(L \leq x) = P(\mu + \sigma Z \leq x) = P(z \leq \frac{x-\mu}{\sigma}) = \Phi(\frac{x-\mu}{\sigma}),$$

Hence,

$$F_L(x) = \alpha \implies \Phi(\frac{x-\mu}{\sigma}) = \alpha \implies$$
$$Var_\alpha(L) = x = \mu + \sigma\Phi^{-1}(\alpha).$$

Example

Assume that the loss distribution follows a location scale family i.e.

$$L = \mu + \sigma X, \;\; \mu, \sigma \in \mathbb{R}, \quad X \sim F_X.$$

Then,

$$VaR_\alpha(L) = \mu + \sigma F_X^{-1}(\alpha).$$

An example is the Student distribution.

Example

Assume that the loss distribution is the lognormal distribution

$$\ln X = \mu t + \sigma \sqrt{t} Z, \;\; Z \sim N(0,1).$$

Find $VaR_\alpha(L)$.

We have that

$$F_X(x) = P(X \leq x) = P(\mu t + \sigma\sqrt{t} Z \leq \ln x) = \Phi(\frac{\ln x - \mu t}{\sigma\sqrt{t}}),$$

Hence,

$$F_X(x) = \alpha \implies \Phi(\frac{\ln x - \mu t}{\sigma\sqrt{t}}) = \alpha \implies$$

$$\ln x = \mu\, t + \sigma\, \sqrt{t}\Phi^{-1}(\alpha) \implies Var_\alpha(X) = x = \exp(\mu\, t + \sigma\, \sqrt{t}\Phi^{-1}(\alpha)).$$

Example

A firm has an investment portfolio consisting of 2 assets

Asset	Investment	Daily volatility
1	100000	0.7%
2	400000	0.2%

and the correlation coeffient between the assets is $\rho = 0.8$.
Estimate the $VaR_{99\%}$ for a this position if held for 15 days, using the normal approximation.

The daily variance of the portfolio is

$$\begin{aligned} Var(V) &= (10000 \times \frac{0.7}{100})^2 + (400000 \times \frac{0.2}{100})^2 \\ &+ 2 \times 0.8 \times \left(10000 \times \frac{0.7}{100}\right) \times \left(400000 \times \frac{0.2}{100}\right) \\ &= 700^2 + 800^2 + 2 \times 0.8 \times 700 \times 800 = 2026000, \end{aligned}$$

hence $\sigma_V = 1423.4$.

Hence the volatility of the position for 15 days is $\sigma_{V,N} = \sqrt{15} \times \sigma_V = 5512.8$ and

$$VaR = \Phi^{-1}(0.99)\sigma_{V,N} = 2.3263 \times \sigma_{V,N} \simeq 12824.$$

Example

A pension scheme has an investment in bonds in total of 5000000 euros.

The total modified duration of the portfolio is 3.8 years.

Assuming parallel displacements of the yield curve with daily volatility 0.08% find the $VaR_{95\%}$ of this position if held for 20 days assuming the normal approximation and given that $\Phi^(0.95) = 1.6449$.

We have $\frac{\Delta P}{P} = -D\Delta y$ hence the daily volatity of the return of the portfolio per euro will be $\sigma_R = 3.8 \times 0.08 = 0.304\%$, therefore for the total portfolio $\sigma_{\Delta P} = P\sigma_R = 5000000 \times 0.304\% = 15200$.

For the 20 days position $\sigma_{\Delta P,N} = \sqrt{N}\sigma_{\Delta P} = \sqrt{20} \times 15200 = 67976.5$ hence

$$VaR = \sigma_{\Delta P,N} \times 1.64495 = 112260.4.$$

Example

An investor has a position on an option on an index which is currently at 2500 units. The annual volatility of the index is 19% and the Δ of the option for the current value of the index is 0.56. Estimate the value at risk for this position for $\alpha = 95\%$, if held for 10 days.

The volatility for the underlying, for the 10 days position is

$$\sigma_{\Delta S/S} = \frac{19}{100} \frac{1}{\sqrt{252}} \sqrt{10} = 0.0378,$$

The *VaR* for the derivative portfolio is

$$VaR_{\alpha} = 1.645 \times 0.56 \times 2500 \times 0.0378 = 87.0534$$

VaR and simulation

Simulation is often used to calculate VaR for complex portfolios.

An important concept is that of risk factors Z and the risk mapping $L_i = f_i(t, Z)$ for the various assets in the portfolio.

1. From historical data or parametric models generate scenaria (samples) concerning the changes in the risk factors ΔZ.
2. Use the risk mappings f_i to calculate the effect of the above changes on the assets

$$Y_i = f_i(t, Z(t)) \mapsto Y_i + \Delta Y_i(t) = f_i(t + \Delta t, Z + \Delta Z(t))$$

3. Generate a sample of the new portfolio values

$$V(t, \theta) = \sum_{i=1}^{N} \theta_i Y_i \;\mapsto\; V(t + \Delta, \theta) = \sum_{i=1}^{N} \theta_i (Y_i + \Delta Y_i(t))$$

   and their changes

$$\Delta V = V(t + \Delta t, \theta) - V(t, \theta),$$

4. Use the empirical distribution of ΔV to calculate the quantiles, i.e., VaR.

Example

Consider a portfolio of liabilities at times T_j, $j = 1, \cdots, N$.

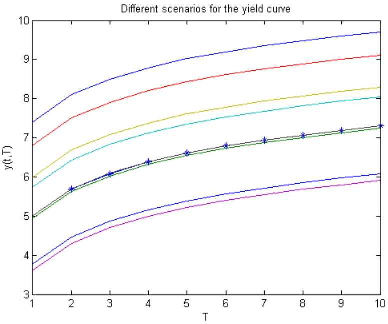
The risk factors Z are the bond yields $y(t, T_i)$ and the risk mapping is

$$L(t, \theta) = \sum_{j=1}^{N} \frac{\theta_j}{(1 + y(t, T_j))^{T_j}},$$

Assuming that the yield curve is today at

$$[y(t, T_j,\ j = 1, \cdots, N] = [5.00, 5.69, 6.09, 6.38, 6.61, 6.79, 6.94, 7.07, 7.19, 7.30]\%$$

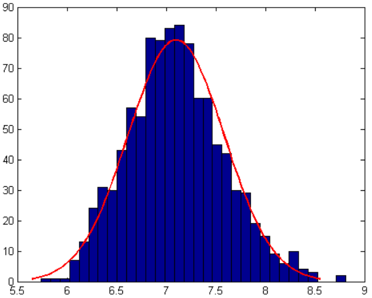
and that we expect fluctuations of the yield curve (parallel) of the form $\Delta y \sim N(0, \sigma^2)$ generate scenarios for the evolution of the portfolio



Scenarios concerning the yield curve

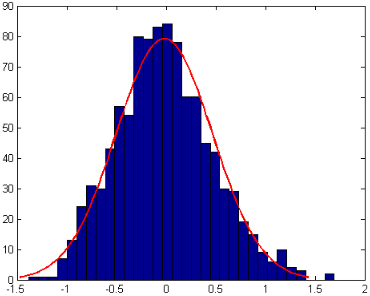
For each of these scenarios we evaluate the value of the portfolio

$$L(t+\Delta t, \theta) = \sum_{j=1}^{N} \frac{\theta_j}{(1+y(t,T_j)+\Delta y)^{T_j}},$$



as well as for the portfolio variation

$$\Delta L = L(t+\Delta t, \theta) - L(t, \theta)$$



Is value at risk the end of the story?

Certainly not, value at risk is a useful risk measure but in many cases can lead to erroneous results especially if the risks are not in the class of elliptical distributions.

Example

Consider defaultable bonds with return b in the case of non default and each with default probability p issued by different issuers and issued at price 100 euros

Using value at risk compare the risk of two portfolios:

- Portfolio A: consisting of N bonds from the same issuer.
- Portfolio B: consisting of N bonds, each by a different issuer.

Let Y_i be the indicator random variable of default for issuer i.

The loss for bond i is

$$L_i = 100Y_i - 100b(1 - Y_i) = \begin{cases} 100 & \text{default} \\ -100b & \text{non default} \end{cases}$$

where negative loss means profit.

Then,

$$L_A = NL_1 = N(100(1+b)Y_1 - 100b),$$

$$L_B = \sum_{i=1}^{N} L_i = 100(1+b)\sum_{i=1}^{N} Y_i - 100Nb$$

Since $Y_i \in Ber(p)$ and Y_i, i.i.d.

$$L_A = 100N(1+b)Y_1 - 100Nb, \quad Y_1 \sim Ber(p),$$
$$L_B = 100(1+b)Z - 100Nb, \quad Z \sim B(N,p).$$

For example if $N = 100$, $b = 5\%$ and $p = 2\%$ a numerical calculation yields that

$$VaR_{95\%}(L_A) \leq VaR_{95\%}(L_B)$$

which is nonsensical from the economic point of view.

Alternatively,

$$VaR_\alpha(\sum_{i=1}^{100} L_i) \geq 100 VaR_\alpha(L_i),$$

i.e. VaR is not subadditive, hence not providing incentives for differentiation!

Tail VaR, Average VaR, Expected shortfall

Tail VaR

$$TVaR_\alpha(L) = \mathbb{E}[L \mid L > VaR_\alpha(L)],$$

Average VaR

$$ES_\alpha(L) = AVaR_\alpha(L) = \frac{1}{1-\alpha}\int_\alpha^1 VaR_\gamma(L)d\gamma.$$

$$ES_\alpha(L) = AVaR_\alpha(L) = \lambda_\alpha\, TVaR_\alpha(L) + (1-\lambda_\alpha)\, VaR_\alpha(L),$$
$$\lambda_\alpha = \frac{1 - P(L \leq VaR_\alpha(L))}{1-\alpha}.$$

For continuous distributions $\lambda_\alpha = 1$, hence

$$TVaR_\alpha(L) = ES_\alpha(L) = AVaR_\alpha,$$

In general

$$TVaR_\alpha(L) \geq AVaR_\alpha(L) \geq VaR_\alpha(L),$$

Example

Suppose that $L \sim N(\mu, \sigma^2)$.

Calculate $AVaR_\alpha$ and $TVaR_\alpha$.

Since the distribution is continuous

$$AVaR_\alpha(L) = ES_\alpha(L) = TVaR_\alpha(L).$$

Then,

$$\begin{aligned} AVaR_\alpha(L) &= ES_\alpha(L) = TVaR_\alpha(L) \\ &= \frac{1}{1-\alpha}\int_\alpha^1 (\mu + \sigma\Phi^{-1}(\gamma))d\gamma \\ &= \mu + \sigma ES_\alpha(Z), \end{aligned}$$

where $Z \sim N(0,1)$.

$$
\begin{aligned}
AVaR_\alpha(Z) &= ES_\alpha(Z) = TVaR_\alpha(Z) \\
&= \frac{1}{1-\alpha}\int_\alpha^1 \Phi^{-1}(\gamma)d\gamma \\
&= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty x\Phi'(x)dx = \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^\infty x\phi(x)dx,
\end{aligned}
$$

where we performed the change of variables $\gamma = \Phi(x)$ and used the notation $\phi(x) = \Phi'(x)$.

Since

$$
x\phi(x) = \phi'(x),
$$

we have

$$
AVaR_\alpha(Z) = ES_\alpha(Z) = TVaR_\alpha(Z) = \frac{1}{1-\alpha}\phi(\Phi^{-1}(\alpha)).
$$

Convex risk measures

Definition

Let $\mathbb{X}$ be the set of all risks. The mapping $\rho : \mathbb{X} \to \mathbb{R}$ is called a convex risk measure if:

1. For every $L \in \mathbb{X}$ and $x \in \mathbb{R}$ we have $\rho(L + x) = \rho(L) + x$.
2. For every $L_1, L_2 \in \mathbb{X}$ such that $L_1 \leq L_2$ a.s. it holds that $\rho(L_1) \leq \rho(L_2)$.
3. For every $L_1, L_2 \in \mathbb{X}$, $\lambda \in [0, 1]$, we have that $\rho(\lambda L_1 + (1 - \lambda)L_2) \leq \lambda\rho(L_1) + (1 - \lambda)\rho(L_2)$.

AVaR and TVaR are convex risk measures.