Financial Mathematics

Option pricing in multiple periods

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We now consider option pricing in multiple periods

We start with the binomial model and develop an algorithm for the determination of the pricing function

We then provide a general proof using martingales of the pricing formula valid for any discrete model.

Finally we provide a pricing function for the continuous time model, leading to the celebrated Black-Schoes formula

Pricing in a general market model using martingalew

We will show the general pricing formula

$$P(t,\omega) = \frac{1}{(1+r)^{T-n}} \mathbb{E}_Q[F(S(T) \mid \mathcal{F}_n],$$
$$\mathcal{F}_n = \sigma(S_0, S_1, \cdots, S_n)$$

for any market model, as long as it supports an equivalent martingale measure Q i.e. a measure such that

$$S_n^* = \mathbb{E}_Q[S_{n+1}^* \mid \mathcal{F}_n], \;\; S_n^* = \frac{1}{(1+r)^n} S_n.$$

We will use the idea of replicating portfolio, i.e. a portfolio that at expiry T exactly reproduces the option, i.e. $V(T) = F(S(T))$.

This portfolio will be chosen to be self-financing and by absense of arbitrage it should have exactly the same properties as the option for any other time $t < T$, hence, its value in the market will provide the price of the option at time t.

Same principle as for the single period case, but a bit more involved: The portfolio is rebalanced every time period from 0 till T!

The portfolio consists of positions θ_0 in the riskless asset B_0 with return r and θ_1 in the risky asset S.

At time n^- the investor chooses a portfolio $(\theta_0(n), \theta_1(n))$ with which he/she enters the market at time n and keeps it until time $(n+1)^-$, where it is updated to $(\theta_0(n+1), \theta_1(n+1))$ etc

Important note: $\theta_0(n), \theta_1(n)$ are known by the history of the market up to time $n-1$ i.e.

$$(\theta_0(n), \theta_1(n)) \in \mathcal{F}_{n-1}, \;\; \forall\, n \;\text{ predictable processes}$$

Just before the market opens at $(n+1)^-$ the value of this portolio is:

$$V(n+1)^- = \theta_0(n+1)B_0(n) + \theta_1(n+1)S(n,\omega)$$

Just when the market opens at $(n+1)^+$ the value of this portfolio is:

$$V(n+1)^+ = \theta_0(n+1)B_0(n+1) + \theta_1(n+1)S(n+1,\omega)$$

Self financing portfolio:

$$V(n+1)^- = V(n)^+$$

You will not inject extra cash into the portfolio!

$$\begin{aligned} & (\theta_0(n+1) - \theta_0(n))B_0(n) + (\theta_1(n+1) - \theta_1(n))S(n,\omega) = 0 \\ \iff \quad & (\theta_0(n+1) - \theta_0(n)) + (\theta_1(n+1) - \theta_1(n))S^*(n,\omega) = 0, \quad (*) \end{aligned}$$

where $S^*(m,\omega) := \frac{1}{B_0(n)}S(n,\omega)$ (discounting).

This means that to change our portfolio between times n^+ and $(n+1)^-$, we simly reallocate our net gains from the investment to the two assets – no extra cash injection!

Discount at times $(n+1)^+$ and n^+:

$$V^*(n+1) = \frac{1}{B_0(n+1)} V(n+1) = \theta_0(n+1) + \theta_1(n+1) S^*(n+1),$$
$$V^*(n) = \frac{1}{B_0(n)} V(n) = \theta_0(n) + \theta_1(n) S^*(n).$$

and subtracting and using the self-financing condition (*):

$$\begin{aligned} V^*(n+1) - V^*(n) &= (\theta_0(n+1) - \theta_0(n)) + \theta_1(n+1) S^*(n+1) - \theta_1(n) S^*(n) \\ &= -(\theta_1(n+1) - \theta_1(n)) S^*(n,\omega) + \theta_1(n+1) S^*(n+1) - \theta_1(n) S^*(n) \\ &= \theta_1(n+1)(S^*(n+1) - S^*(n)) \end{aligned}$$

Taking $\mathbb{E}_Q[\cdot \mid \mathcal{F}_n]$ and recalling that $S^*(n)$ is a Q-martingale:

$$\begin{aligned} \mathbb{E}_Q[V^*(n+1) \mid \mathcal{F}_n] - V^*(n) &= \mathbb{E}_Q[\theta_1(n+1)(S^*(n+1) - S^*(n)) \mid \mathcal{F}_n] \\ &= \theta_1(n+1)\mathbb{E}_Q[(S^*(n+1) - S^*(n)) \mid \mathcal{F}_n] = 0, \end{aligned}$$

hence

$$\mathbb{E}_Q[V^*(n+1) \mid \mathcal{F}_n] = V^*(n)$$

hence the discounted value of any self-financing portfolio is a Q-martingale (true for any market process!)

Let $(\bar{\theta}_0, \bar{\theta}_1)$ be a self-financing portfolio that replicates the option, i.e. such that the corresponding value process satisfies $\bar{V}(T) = F(S(T))$.

Then

$$\begin{aligned}
\bar{V}^*(T-1) &= \mathbb{E}_Q[\bar{V}^*(T) \mid \mathcal{F}_{T-1}] = \mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_{T-1}], \\
\bar{V}^*(T-2) &= \mathbb{E}_Q[\bar{V}^*(T-1) \mid \mathcal{F}_{T-2}] \\
&= \mathbb{E}_Q[\mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_{T-1}] \mid \mathcal{F}_{T-2}] \\
&= \mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_{T-2}]
\end{aligned}$$

and by induction

$$\bar{V}^*(n) = \mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_n], \ \forall\, n$$

By the absense of arbitrage $\bar{V}(n) = P(n)$ (i.e. equals the price of the option) hence:

$$P(n) = (1+r)^{-(T-n)}\mathbb{E}_Q[F(S(T)) \mid \mathcal{F}_n], \ \forall\, n$$

which is the pricing formula!

Note that we do not need to know what the replicating portfolio is for the pricing: We only need to assume that such a portfolio exists (recall completeness).

Moreover,

$$\begin{aligned}\mathbb{E}_Q[P(n+1) \mid \mathcal{F}_n] &= \mathbb{E}_Q[\mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n] \\ &= \mathbb{E}_Q[F^*(S(T)) \mid \mathcal{F}_n] = P(n),\end{aligned}$$

i.e. $P(n)$ is the best prediction of the price $P(n+1)$ under the EMM!

If the model has the Markov property then we do not need the full information set $\mathcal{F}_n = \sigma(S_0, S_1, \cdots, S_n)$ but only $\sigma(S_n)$!

Hence,

$$\begin{aligned}P(n) &= \frac{1}{1+r}\mathbb{E}_Q[P(n+1) \mid \mathcal{F}_n] = \frac{1}{1+r}\mathbb{E}_Q[P(n+1) \mid \sigma(S_n)], \\ P(T) &= F(S(T))\end{aligned}$$

i.e. you can recover the price by "myopically" iterating backwards this price equation in time.

The RHS depends on the state of the market at time n.

This equation is the basis for a numerical algorithm for option pricing.

Application: The binomial model and the binomial tree algorithm

$$S(n+1) = H(n)S(n), \quad Q(H(n) = u) = q = \frac{1+r-d}{u-d} = 1 - Q(H(n) = d)$$

This model has the Markov property so that for any function F, there exists a function $V(t, s)$ such that

$$\mathbb{E}_Q[F(S(T)) \mid \mathcal{F}_n] = V(n, S(n))$$

For that

$$\begin{aligned} P(n) &= \frac{1}{1+r}\mathbb{E}_Q[P(n+1) \mid \mathcal{F}_n] = \frac{1}{1+r}\mathbb{E}_Q[P(n+1) \mid \sigma(S_n)], \\ P(T) &= F(S(T)) \end{aligned}$$

becomes

$$\begin{aligned} V(n, S(n)) &= \frac{1}{1+r}\mathbb{E}_Q[V(n+1, S(n+1)) \mid \sigma(S(n))] \\ V(T, S(T)) &= F(S(T)) \end{aligned} \tag{1}$$

If $S(n) = S(0)u^k d^{n-k}$ then

$$S(n+1) = \begin{cases} S(0)u^{k+1}d^{n-k} & \text{with prob.} \quad q \\ S(0)u^k d^{n+1-k} & \text{with prob.} \quad 1-q \end{cases}$$

so that

$$\begin{aligned} &\mathbb{E}_Q[V(n+1, S(n+1)) \mid S(n) = S(0)u^k d^{n-k}] \\ &= qV(n+1, S(0)u^{k+1}d^{n-k}) + (1-q)V(n+1, S(0)u^k d^{n+1-k}) \end{aligned}$$

We use the simpler notation $V(n, S_0 u^k d^{n-k}) =: P(n,k)$ and then (1) becomes

$$\begin{aligned} P(n,k) =& \frac{1}{1+r}\Big(qP(n+1,k+1) + (1-q)P(n+1,k)\Big), \quad k = 0, \cdots, n, \\ & \qquad\qquad n = 0, \cdots, T-1, \\ P(T,k) =& F(S_0 u^k d^{T-k}), \quad k = 0, \cdots, T. \end{aligned} \tag{2}$$

For each time instant n we consider $k = 0, 1, \cdots, n$ states:

$$(n,k) \quad \begin{matrix} \nearrow & (n+1, k+1) \\ \\ \searrow & (n+1, k) \end{matrix}$$

Example

Find the price of a call option that matures in 3 months.
The underlying is a stock with annual volatility $\sigma = 0.169$ and the price of the stock today is $S(0) = 10$, whereas the strike of the option is $K = 9.8$.
The riskless rate is $r = 5\%$ p.a.

We need to fit a binomial model to these data: Let us split the 3 month period into 3 subintervals $\delta t = 1 mo = \frac{1}{12} ye$ each.

The binomial model parameters will be

$$\begin{aligned} u &= \exp(\sigma\sqrt{\delta t}) = \exp(0.169\sqrt{\frac{1}{12}}) = 1.05, \\ d &= \frac{1}{u} \simeq 0.95 \end{aligned}$$

The risk neutral probability is

$$q = \frac{1 + r\delta t - d}{u - d} \simeq \frac{e^{r\delta t} - d}{u - d} = 0.53$$

(r is a rate so the actual return in a period of δt will be $r\delta t$)

To find the price we run the binomial tree algorithm for $T = 2$ and for the above parameters:

Possible prices for the underlying:

$$\begin{aligned} t = 0, &\qquad S_0 = S_0^0 = 10 \\ t = 1, &\qquad S_1^0 = 9.5, \ \text{ή} \ S_1^1 = 10.5 \\ t = 2, &\qquad S_2^0 = 9.025, \ \text{ή} \ S_2^1 = 9.975, \ \text{ή} \ S_2^2 = 11.025 \end{aligned}$$

We use the notation V_n^k instead of $P(n, k)$ for simplicity:

At $n = T$:

$$V_2^0 = 0, \ \text{ή} \ V_2^1 = 0.175, \ \text{ή} \ V_2^2 = 1.225$$

We now proceed to $n = 1$, where we must consider $k = 0$ and $k = 1$:

$$V_1^1 = \frac{1}{1 + r\delta t}(qV_2^2 + (1 - q)V_2^1) \simeq 0.73$$

and

$$V_0^1 = \frac{1}{1 + r\delta t}(qV_2^1 + (1 - q)V_2^0) \simeq 0.0924$$

We next proceed to $n = 0$ where we must consider $k = 0$ only:

$$V_0^0 = \frac{1}{1 + r\delta t}(qV_1^1 + (1 - q)V_1^0) \simeq 0.43$$

Hedging and the greeks

How do you hedge the risk that you undertake when entering a derivative contract?

Suppose you sell (short position) a call option: You have the obligation of supplying one piece of the underlying at T at price K (which has to be acquired at $S(T) > K$).

How do you make sure that you can cover this position (hedge the risk)?

Keep one piece of the underlying in your portfolio for the whole period: If the buyer exercises simply hand him/her over the stock.

Not a very clever strategy because you lose if the stock depreciates (and the option is not exercised).

Must find alternative strategies: Delta hedging.

Seller of the option has a position -1 in the option

At the same time keeps a position Δ in the underlying S

Total value of portfolio $\Pi = \Delta \times S - V$ where V is the value of the option

V is correllated to S: Can we choose Δ so that Π is insensitive to the fluctuations of S?

This Δ must change dynamically with the state of the market

Hedging!

At state (n, j), choose hedge position Δ_n^j and keep it up to time $n+1$ where it will be further updated

$$\begin{array}{rll} \text{Time } \; n & \Pi_n^j = \Delta_n^j \times S_n^j - V_n^j & \\ \text{Time } \quad n+1 & \Pi_{n+1}^j = \Delta_{n+1}^j \times S_{n+1}^j - V_{n+1}^j & \text{Down} \\ & \Pi_{n+1}^{j+1} = \Delta_{n+1}^{j+1} \times S_{n+1}^{j+1} - V_{n+1}^{j+1} & \text{Up} \end{array}$$

Choose Δ_n^j so that

$$\Pi_{n+1}^j = \Pi_{n+1}^{j+1} \iff \Delta_n^j = \frac{V_{n+1}^{j+1} - V_{n+1}^j}{S_{n+1}^{j+1} - S_{n+1}^j}$$

so that Π is riskless.

No arbitrage implies that Π has return r,

$$\Pi_{n+1}^{j+1} = \Pi_{n+1}^j = (1+r)\Pi_n^j \iff V_n^j = \frac{1}{1+r}(qV_{n+1}^{j+1} + (1-q)V_{n+1}^j)$$

where we also used $S_{n+1}^{j+1} = uS_n^j$, and $S_{n+1}^j = dS_n^j$.

$$\Delta_n^j = \frac{V_{n+1}^{j+1} - V_{n+1}^{j}}{S_{n+1}^{j+1} - S_{n+1}^{j}}$$

is called the Δ-hedging portfolio – it is the position you must keep in the underlying along with your position in the derivative in order to hedge the risk you undertake by entering the derivative position

Δ is the sensitivity of the value of the option with respect to the changes in the value of the underlying

Key quantity in risk management

Can easily be calculated on the binomial tree

Example

Find the price and the Δ of a call option that matures in 3 months.
The underlying is a stock with annual volatility $\sigma = 0.169$ and the price of the stock today is $S(0) = 10$, whereas the strike of the option is $K = 9.8$.
The riskless rate is $r = 5\%$ p.a.

We need to fit a binomial model to these data: Let us split the 3 month period into 3 subintervals $\delta t = 1mo = \frac{1}{12}ye$ each.

The binomial model parameters will be

$$\begin{aligned} u &= \exp(\sigma\sqrt{\delta t}) = \exp(0.169\sqrt{\frac{1}{12}}) = 1.05, \\ d &= \frac{1}{u} \simeq 0.95 \end{aligned}$$

The risk neutral probability is

$$q = \frac{1 + r\delta t - d}{u - d} \simeq \frac{e^{r\delta t} - d}{u - d} = 0.53$$

(r is a rate so the actual return in a period of δt will be $r\delta t$)

To find the price we run the binomial tree algorithm for $T = 2$ and for the above parameters:

Possible prices for the underlying:

$$
\begin{aligned}
t = 0, &\qquad S_0 = S_0^0 = 10 \\
t = 1, &\qquad S_1^0 = 9.5, \;\text{ή}\; S_1^1 = 10.5 \\
t = 2, &\qquad S_2^0 = 9.025, \;\text{ή}\; S_2^1 = 9.975, \;\text{ή}\; S_2^2 = 11.025
\end{aligned}
$$

We use the notation V_n^k instead of $P(n, k)$ for simplicity:

At $n = T$:

$$V_2^0 = 0, \;\text{ή}\; V_2^1 = 0.175, \;\text{ή}\; V_2^2 = 1.225$$

We now proceed to $n = 1$, where we must consider $k = 0$ and $k = 1$:

$$V_1^1 = \frac{1}{1 + r\delta t}(qV_2^2 + (1-q)V_2^1) \simeq 0.73$$

$$\Delta_1^1 = \frac{V_2^2 - V_2^1}{S_2^2 - S_2^1} = \frac{1.225 - 0.175}{11.025 - 9.975} = 0.999$$

and

$$V_1^0 = \frac{1}{1 + r\delta t}(qV_2^1 + (1-q)V_2^0) \simeq 0.0924$$

$$\Delta_1^0 = \frac{V_2^1 - V_2^0}{S_2^1 - S_2^0} = \frac{1.175 - 0}{9.975 - 9.025} = 0.184$$

We next proceed to $n = 0$ where we must consider $k = 0$ only:

$$V_0^0 = \frac{1}{1 + r\delta t}(qV_1^1 + (1-q)V_1^0) \simeq 0.43$$

$$\Delta_0^0 = \frac{V_1^1 - V_1^0}{S_1^1 - S_1^0} = \frac{0.73 - 0.0924}{10.5 - 9.5} = 0.6376$$

Monte Carlo pricing

Monte-Carlo pricing exploits the representation

$$P(n) = \mathbb{E}_Q[(1+r)^{-(T-n)} F(S(T)) \mid \mathcal{F}_n]$$

and the Markov property for the underlying prices to calculate the price.

- Start at $S_n = S$
- For $j = 1, \cdots, M$
  - Iterate

$$\begin{aligned} & S(k+1)^j = H(k+1)^j S(k)^j, \;\; k = n, \cdots, T-1, \\ & S(n)^j = S, \\ & \{H(k)^j \;:\; k = n, \cdots, T-1\} \;\; i.i.d. \;\; Prob(H(k)^j) = q \end{aligned}$$

- Collect a sample for $S(T)^j$ for $j = 1, \cdots, M$
- Transform it to a sample for $F(S(T)^j)$ for $j = 1, \cdots, M$
- Calculate the sample mean

$$P(n) = (1+r)^{-(T-n)} \frac{1}{M} \sum_{j=1}^{M} F(S(T)^j)$$

The Black-Scholes model

We recall the continuous limit of the binomial model

$$S(t) = S(0) \exp((\mu - \sigma^2/2)t + \sigma W(t)),$$

where $\{W(t) \ : \ t \geq 0\}$ is the Wiener process, i.e., a family of random variables such that

- $W(t_2) - W(t_1) \sim N(0, t_2 - t_1)$, for all $t_2 \geq t_1 \geq 0$
- $W(t_3) - W(t_2)$ and $W(t_2) - W(t_1)$ independent random variables for all $t_3 \geq t_2 \geq t_1 \geq 0$
- $t \mapsto W(t)$ is a continuous random function
- $W(0) = 0$ a.s.

For this model the log return for the underlying follows the normal distribution (i.e. the underlying follows the lognormal distribution)

Two parameters in the model:

- μ mean log return
- σ volatility: the variance of log returns

For this model it is easy to show that

$$S(t_2) = S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1) + \sigma(W(t_2) - W(t_1)))$$

so from the properties of the Wiener process

$$\begin{aligned}
\mathbb{E}[S(t_2) \mid \mathcal{F}_{t_1}] &= \mathbb{E}[S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1) + \sigma(W(t_2) - W(t_1))) \mid \mathcal{F}_{t_1}] \\
&= S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1))\mathbb{E}[\exp(\sigma(W(t_2) - W(t_1))) \mid \mathcal{F}_{t_1}] \\
&= S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1))\mathbb{E}[\exp(\sigma(W(t_2) - W(t_1)))] \\
= S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1))&\frac{1}{\sqrt{2\pi(t_2 - t_1)}}\int_{-\infty}^{\infty}\exp(\sigma z)\exp\left(-\frac{z^2}{2(t_2 - t_1)}\right)dz \\
&S(t_1)\exp((\mu - \sigma^2/2)(t_2 - t_1))\exp(\sigma^2(t_2 - t_1) \\
&= S(t_1)\exp(\mu(t_2 - t_1)).
\end{aligned}$$

To price the option we need an EMM, i..e, a measure Q under which

$$\mathbb{E}_Q[e^{-rt_2}S(t_2) \mid \mathcal{F}_{t_1}] = exp(-rt_1)S(t_1).$$

From the above we can see that passing to the EMM, will have the effect of setting $\mu = r$ in the model

Of course this is an oversimplification and requires a deep result from stochastic analysis: Girsanov's theorem

We now apply the pricing formula

$$
\begin{aligned}
P(t) = \mathbb{E}_Q[F(S(T)) \mid \mathcal{F}_t] &= \mathbb{E}[F(S(0)\exp((r - \sigma^2/2)t + \sigma W(t)) \mid \mathcal{F}_t] \\
&= \mathbb{E}[F(S(t)\exp((r - \sigma^2/2)(T - t) + \sigma(W(T) - W(t))) \mid \mathcal{F}_t] \\
&= \mathbb{E}[F(S(t)\exp((r - \sigma^2/2)(T - t) + \sigma Z)] \;\; Z \sim N(0, T - t)
\end{aligned}
$$

$$
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T - t)}} F(S(t)\exp((r - \sigma^2/2)(T - t) + \sigma z)\exp\left(-\frac{z^2}{2(T - t)}\right) dz
$$

For

- $F(s) = (s - K)^+$ (call option) or
- $F(s) = (K - s)^+$ (put option)

the above pricing formula can be calculated analytically.

- Call option $F(s) = (s-K)^+$, $S(t) = S$.

$$P(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} (S\exp((r-\sigma^2/2)(T-t)+\sigma z) - K)^+ \exp\left(-\frac{z^2}{2(T-t)}\right) dz$$
$$= S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where Φ is the normal cdf

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(T-t))}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(S/K) + (r-\sigma^2/2)(T-t))}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

- Put option $F(s) = (K-s)^+$, $S(t) = S$.

$$P(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)}} (K - S\exp((r-\sigma^2/2)(T-t)^+ + \sigma z) \exp\left(-\frac{z^2}{2(T-t)}\right) dz$$
$$= Ke^{-r(T-t)}\Phi(-d_2) - S\Phi(-d_1)$$

where Φ is the normal cdf

$$d_1 = \frac{\ln(S/K) + (r+\sigma^2/2)(T-t))}{\sigma\sqrt{T-t}},$$
$$d_2 = \frac{\ln(S/K) + (r-\sigma^2/2)(T-t))}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t},$$

Example

Let S be a stock with volatility $\sigma = 0.3$ p.a. Find the price of a European call option on the stock with strike $K = 40$, and expiry $T = 1$ year, if the price of the stock in the market today $t = 0$ is 50 euros.
The riskless rate is 5% p.a.

We will apply the Black scholes formula:

$$\begin{aligned}
d_1 &= \frac{(log(50/40) + (0.5 * 0.3 * 0.3 + 5/100) * 1)}{(0.3 * \sqrt{1})} = 1.0605 \\
d_2 &= 1.0605 - 0.3 * \sqrt{1} = 0.7605 \\
\Phi(d_1) &= 0.8555 \\
\Phi(d_2) &= 0.7765 \\
C(0, 50) &= 50 * 0.8555 - 40 * exp(-5/100) * 0.7765 = 13.2310
\end{aligned}$$

Hence, the price of the call is 13.23 Euros.

This price takes into account the risk neutral probability that the underlying keeps above 40 in a year's time so that the option is exercised.

The price of the option obviously depends on the price of the underlying at t

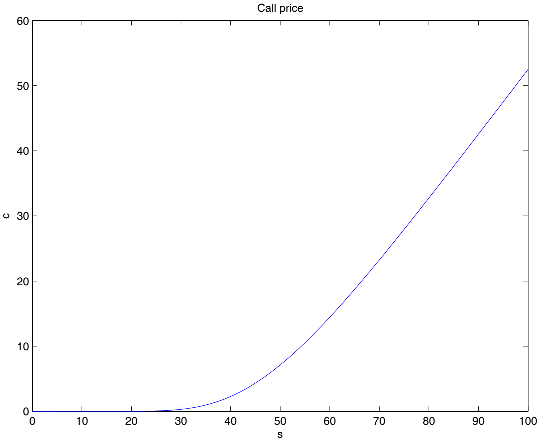


Figure: Call option price with expiry $T = 1$ y, strike $K = 40$ when the volatility of the underlying is $\sigma = 0.3$, and the riskless rate $r = 5\%$ p.a. at $t = 0$ for various values of the underlying.

The value of the option depends on other quantities as well: One of the most important is volatility

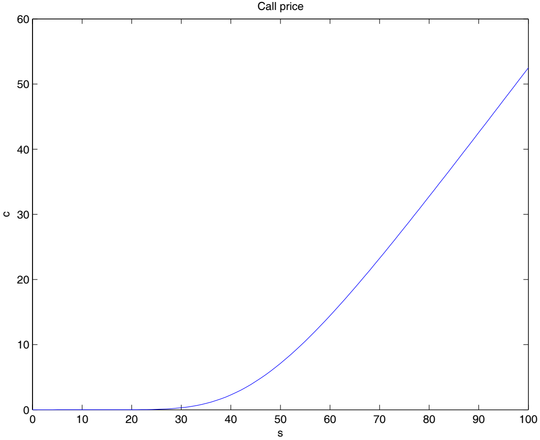


Figure: Dependence of call price on the volatility (all other quantities as above)

Example

Let S be a stock with volatility $\sigma = 0.3$ p.a. Find the price of a European put option on the stock with strike $K = 40$, and expiry $T = 1$ year, if the price of the stock in the market today $t = 0$ is 50 euros.
The riskless rate is 5% p.a.

Using the Black-Scholes formula

$$\begin{aligned} d_1 &= 1.0605 \\ d_2 &= 1.0605 - 0.3 * \sqrt{1} = 0.7605 \\ N(d_1) &= 0.8555 \\ N(d_2) &= 0.7765 \\ P(0, 50) &= 1.2802 \end{aligned}$$

The price of the put is $P = 1.28$, which reflects the expectations (under the risk neutral measure) that the stock price in a year's time falls below the strike $K = 40$, and thus exercise the option.

The value of the option depends on the price of the underlying at the time of pricing

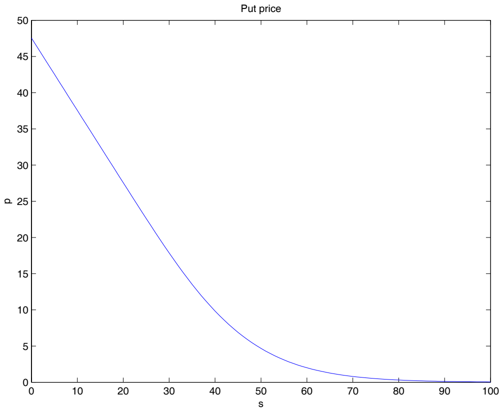


Figure: Put option price with expiry $T = 1$ y, strike $K = 40$ when the volatility of the underlying is $\sigma = 0.3$, and the riskless rate $r = 5\%$ p.a. at $t = 0$ for various values of the underlying.

The value of the option depends on other quantities as well: One of the most important is volatility

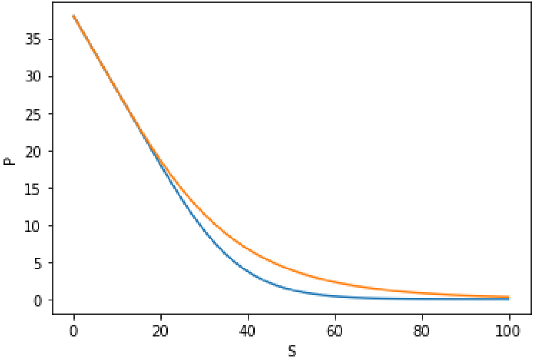


Figure: Dependence of put price on the volatility (all other quantities as above) $\sigma = 0.3$ (blue), $\sigma = 0.5$ (orange)

The greeks in the Black-Scholes model

The greeks are quantities that quantify the sensitivity of the value of the derivative asset with respect to various variables such as the underlying price, the volatility, the riskless rate etc.

These are important for risk management aspects.

Δ

Δ quantifies the sensitivity of the value of the option with respect to the changes in the value of the underlying

It can be interpreted as the position that an investor can take in the underlying so as in conjunction with the short position in the option hedges the risk (fluctuations)

In particular Δ is chose so as to make the portfolio $\Pi = \Delta \times S - V$ locally riskless

For the Black-Scholes model it can be shown that

$$\Delta = \frac{\partial V}{\partial S}$$

Moreover, for this model Δ can be calculated in closed form in terms of

$$\begin{aligned} \Delta_C(t, S) &= \Phi(d_1) \\ \Delta_P(t, S) &= \Phi(d_1) - 1 \end{aligned}$$

Why?

For the BS model the underlying follows the dynamics

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t)$$

Apply Itō's rule to the function $\Pi(t, S) = \Delta \times S - V(t, S)$ recalling the mnemonic rule that $dt^2 \to 0$, $dtdS \to 0$ and $dS^2 \to dt$

Then,

$$\begin{aligned} d\Pi(t, S) &= \Delta dS - \frac{\partial V}{\partial t}(t, S)dt - \frac{\partial V}{\partial S}dS - \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2 \\ &= \Delta(\mu S dt + \sigma S dW) - \frac{\partial V}{\partial S}dt - \frac{\partial V}{\partial S}(\mu S dt + \sigma S dW) - \frac{1}{2}\frac{\partial V}{\partial S^2}\sigma^2 S^2 dt \\ &= \left(\Delta \mu S - \frac{\partial V}{\partial S} - \mu S\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right)dt + \left(\Delta \mu S - \frac{\partial V}{\partial S}\mu S\right)dW \end{aligned}$$

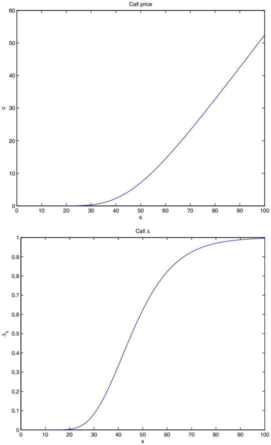
so that choosing $\Delta = \frac{\partial V}{\partial S}$ we have no dW term.

For this choice and by absense of arbitrage $d\Pi = r\Pi dt$ and we get that V solves

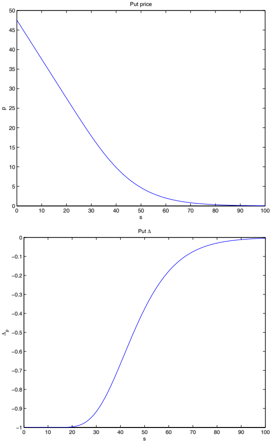
$$\frac{\partial V}{\partial t} - rS\frac{\partial V}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$$

which is the Black-Scholes-Merton PDE.

Δ for a call



Δ for a put



Γ

The hedge Δ changes with S

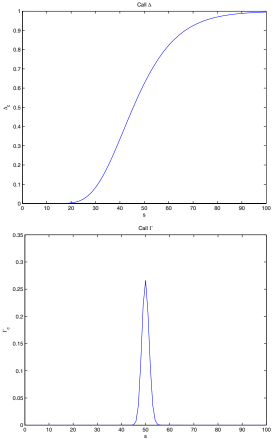
A measure of the sensitivity of Δ with S is Γ:

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V}{\partial S^2}$$

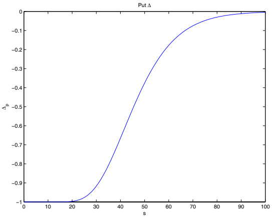
For the Black-Scholes model we can calculate Γ explicitly in closed form:

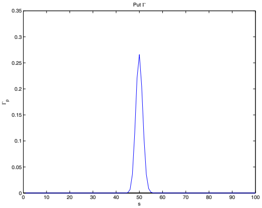
$$\begin{aligned} \Gamma_C &= \frac{\Phi^{'}(d_1)}{S\sigma\sqrt{T-t}} \\ \Gamma_P &= \Gamma_C \\ \Phi^{'}(d_1) &= \frac{1}{\sqrt{2\pi}} exp\left(-\frac{d_1^2}{2}\right) \end{aligned}$$

Call: Δ and Γ



Put: Δ and Γ





Convergence of the binomial model to the Black-Scholes

Fix a horizon $[0, T]$ and get a partition $P_N := \{t_0^N, t_1^N, \cdots, t_N^N\}$ for it in terms of N intervals of length $\Delta t^N = \frac{T}{N}$.

Then, if σ is the volatility of the underlying, construct a binomial model with $\bar{S}_{n+1}^N = H_{n+1}^N \bar{S}_n^N$ with parameters

$$u^N = \frac{1}{d^N} = \exp(\sigma\sqrt{\Delta t^N})$$

and an appropriate choice for the probability.

We have seen that in the limit as $N \to \infty$, the corresponding binomial model converges to the geometric Brownian motion (associated with the Black-Scholes model)

$$S(t) = S(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)\right), \quad t \in [0, T] \tag{3}$$

in the sense that for any $t \in [t_n^N, t_{n+1}^N)$ the output of the binomial model $\bar{S}_n$ converges in distribution to $S(t)$ as $N \to \infty$.

Given that we know how to use the paths of the binomial model to price options by Monte Carlo simulation, can we use this binomial model to approximate the price of the options under the continuous time model?

Set

$$\Delta t^N = \frac{T}{N}, \ u^N = (d^N)^{-1} = \exp(\sigma\sqrt{\Delta t^N}), \ q^N = \frac{e^{r\Delta^T} - d^N}{u^N - d^N}$$

and construct the corresponding binomial model $\bar{S}^N(n)$.

For an option with payoff $F(S(T))$ and expiry T, calculate its price using the binomial model

$$P^N(t) = e^{-r(N-n)\Delta^N}\mathbb{E}[F(\bar{S}^N(N)) \mid \mathcal{F}_n], \ \ t \in [t_n^N, t_{n+1}^N)$$

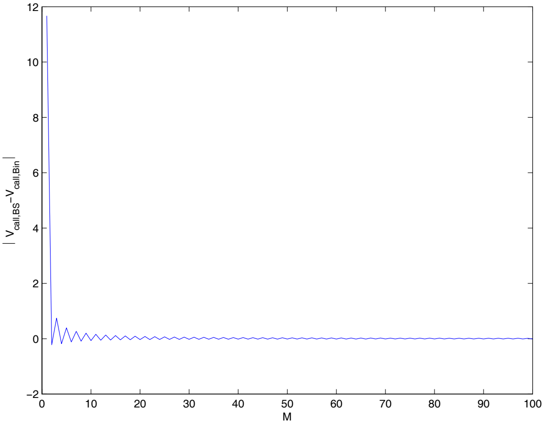
and Monte-Carlo simulation.

As $N \to \infty$ the price $\bar{P}^N(t) \to P(t)$ where $P(t)$ is the corresponding Black-Scholes price for the option i.e.

$$P(t) = \mathbb{E}[e^{-r(T-t)}F(S(T)) \mid \mathcal{F}_t]$$

with $S(t)$ satisfying (3) with $\mu = r$.

This is extremely useful in cases where the option is complicated and the closed form solutions are not available.



centerline
Error of the binomial model approximation of the Black-Scholes price for the European call

Portfolios of options and strategies

In practice we will use not single options but a portfolio of various options with pricing functions V_i, $i = 1, \cdots, k$.

The position in each portfolio is characterized by a real number θ_i

- $\theta_i > 0$ long position (we hold the option)
- $\theta_i < 0$ short position

The value of the portfolio is

$$V = \theta_0 S + \sum_{i=1}^{k} \theta_i V_i$$

which of course depends on the fluctuations of the values of the underlying.

We may find easily the sensitivities (greeks) of the portfolio

$$\Delta = \theta_0 + \sum_{i=1}^{k} \theta_i \Delta_i,$$

$$\Gamma = \sum_{i=1}^{k} \theta_i \Gamma_i$$

By appropriate choice of θ we may make this portfolio Δ or Γ neutral or both! – Standard risk management practices

Example

Let the riskless rate is $r = 5\%$ p.a. and consider a portfolio on an underlying of volatility $\sigma = 0.3$ consisting of

- A short position on a put option of strike $K_1 = 50$
- Two long position on a call option of strike $K_2 = 40$

How could you hedge this portfolio if the value of the stock is in the interval $0 < S < 20$?

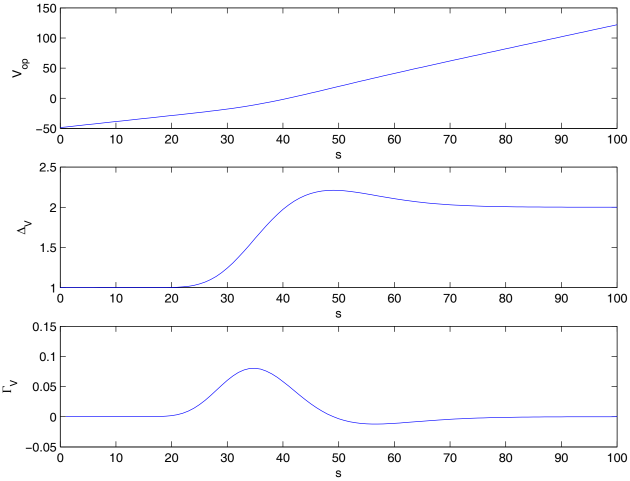
How would your hedging strategy vary if the value of the stock is in the interval $80 < S < 100$?

Choose a position in the underlying θ_0 so that when combined with the position $\theta_1 = -1$ in the put and $\theta_2 = 2$ in the options the net portfolio has a $\Delta \simeq 0$.

This requires choosing θ_0 so that

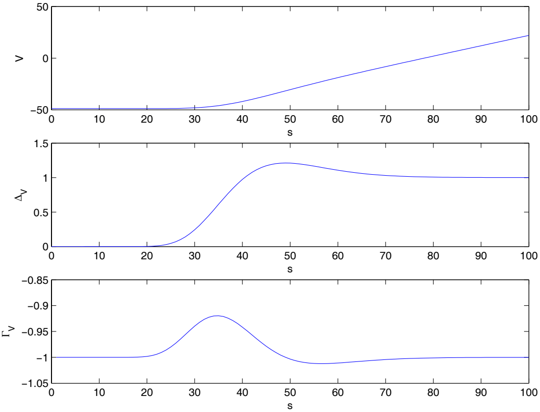
$$\theta_0 = -\theta_1 \Delta_1 - \theta_2 \Delta_2$$

where θ_1, θ_2 as above and Δ_1, Δ_2 can be obtained by the corresponding Black-Scholes formulae.

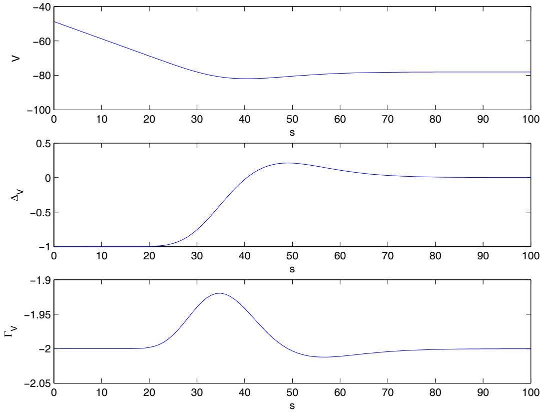


The value and greeks of the derivative portfolio

- In $S \in [0, 20]$ choose $\theta_0 \simeq -1$
- In $S \in [80, 100]$ choose $\theta_0 \simeq -2$



Hedged portfolio for $S \in [0, 20]$



Hedged portfolio for $S \in [80, 100]$

Protective Put

- Long position in the underlying S
- Long position on a put of strike K

Insures the portfolio against losses caused by a fall in the price of S.

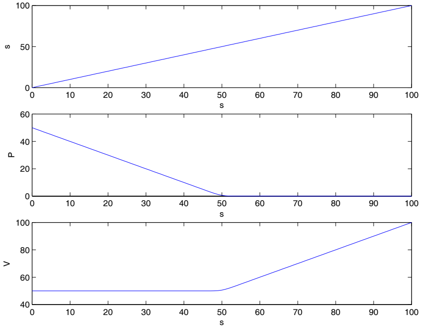


Figure: Payoff of protective put

The price and greeks of this combination can be calculated using e.g. the BS formula

Covered call

- Short position (sell) on a call of strike K
- Long position on the underlying

Insures the portfolio against the risk caused by a rise in the price of S.

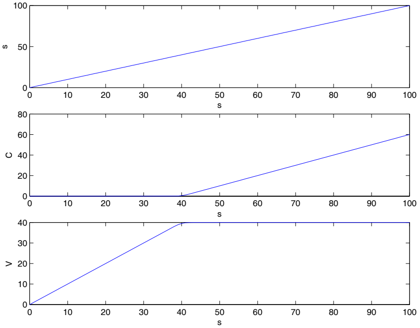


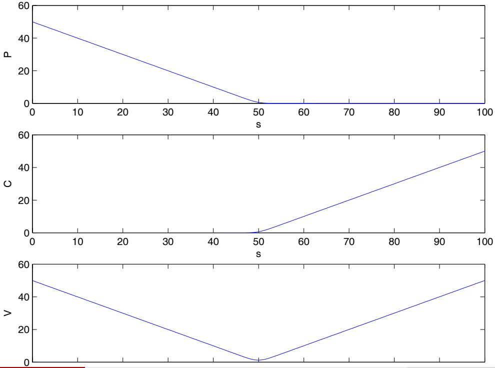
Figure: Payoff of covered call

The price and greeks of this combination can be calculated using e.g. the BS formula

Straddle

Simultaneously buy (or sell) a put and a call on the same underlying with the same characteristics.

Used to speculate on volatilty (no matter if the stock falls or rises around K you win)



Strips and straps

Similar to the above, but with different weighting on the put or call depending on whether you think the market will rise or fall

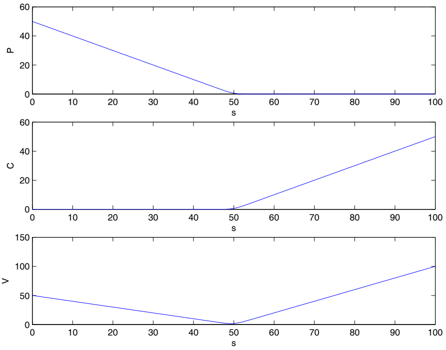


Figure: Strip for a rising market

Spreads: The bear and the bull spread

Sell (short) and buy (long) a derivative with the same characteristics apart from one, e.g. the strike

Bear spread

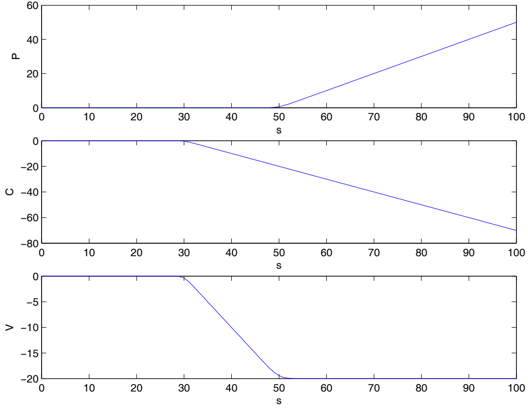


Figure: Payoff for bear spread

Strategy for a falling market

Bull spread

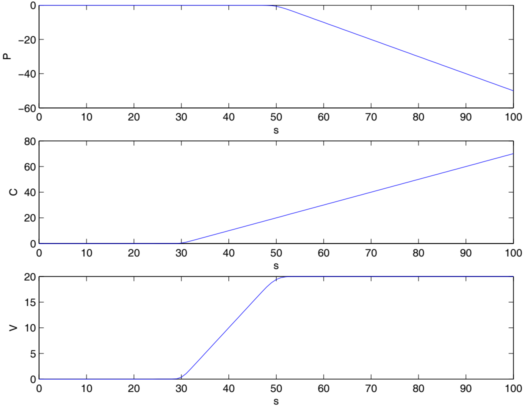


Figure: Payoff for bull spread

Strategy for a falling market

American type options

Definition

An American option is an option that the holder has the right to exercise any time τ before expiry T, providing upon exercise the payoff $F(S(\tau))$.

τ is a random time: $\{\omega \ : \ \tau \leq s\} \in \mathcal{F}_s$ for all s (stopping time)

The price of an american option is

$$P(t) = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}[(1+r)^{T-\tau} F(S(\tau))],$$

where $\mathcal{T}_t$ is the set of stopping times $\tau \geq t$

This is an optimal stopping time the solution τ^* of which will provide the exercise time and the price of the option will be

$$P(t) = \mathbb{E}[(1+r)^{T-\tau^*} F(S(\tau^*))] = \sup_{\tau \in \mathcal{T}_t} \mathbb{E}[(1+r)^{T-\tau} F(S(\tau))]$$

Difficult problems that do not admit closed form solutions

Dynamic programming

For an americal option we need to discriminate between price and payoff:

At any time t:

- Payoff $F(S(t))$ is what you will get if you choose to exercise the option at t
- Price $V(t)$ is what the value of the unexercised option in the market is if you decide to pass it on to somebody else or keep it yourself.

Recall the binomial model and the tree: For each time instant n we

consider $k = 0, 1, \cdots, n$ states:

$$
(n,k) \begin{array}{l} \nearrow \quad (n+1, k+1) \\ \\ \searrow \quad (n+1, k) \end{array}
$$

At (n, k) you

- Either decide to exercise and receive $F(S_n^k)$
- Or keep and treat the option as a European option whose value is

$$V_n^k = \frac{1}{1+r}(qV_n^{k+1} + (1-q)V_n^k)$$

Obviously, you choose the action that provides the best outcome so that

$$V_n^k = \max\left(F(S_n^k), \frac{1}{1+r}(qV_n^{k+1} + (1-q)V_n^k)\right)$$

At expiry $V_T^k = F(S_T^k)$.

The time n^* and state of the market j^* for which

$$(n^*, j^*) = min\{(n, j) : V_n^j = F(S_n^j)\}$$

is the exercise time.

Example

Find the price of an American put and the exercise time on a stock that matures in 3 months with annual volatility $\sigma = 0.169$ and the price of the stock today is $S(0) = 10$, whereas the strike of the option is $K = 9.8$. The price of the stock today is $S(0) = 10$ and the riskless rate is $r = 12\%$ p.a.

As before spilit the 3 month period ($[0, T]$) in 3 subintervals of length $\Delta t = 1mo = 1/12ye$ each.

The binomial model parameters are

$$u = d^{-1} = e^{\sigma\sqrt{\Delta t}} = 1.05 \implies d = 0.95,$$
$$q = \frac{1 + r\Delta t - d}{u - d} = 0.6$$

We must record on any node (n, j) of the tree

- The values of the underlying S_n^j
- The payoffs of the option $F_n^j = (K - S_n^j)^+$ if exercised on the particular node

$$\begin{aligned} t = 0, \;\; S_0 = S_0^0 = 10 \\ t = 1, \;\; S_1^0 = 9.5, \;\; \acute{\eta} \;\; S_1^1 = 10.5 \\ t = 2, \;\; S_2^0 = 9.025, \;\; \acute{\eta} \;\; S_2^1 = 9.975, \;\; \acute{\eta} \;\; S_2^2 = 11.025 \end{aligned}$$

$$\begin{aligned} t = 0, \;\; F_0^0 = 0 \\ t = 1, \;\; F_1^0 = 0.5, \;\; \acute{\eta} \;\; F_1^1 = 0 \\ t = 2, \;\; F_2^0 = 0.975, \;\; \acute{\eta} \;\; F_2^1 = 0.025, \;\; \acute{\eta} \;\; F_2^2 = 0 \end{aligned}$$

To find the option price iterate backwards

$$V_n^j = \max\left(F_n^j, \frac{1}{1+r\Delta t}(qV_{n+1}^{j+1} + (1-q)V_{n+1}^j\right),$$
$$V_T^j = F_T^j$$

For t=1:

- $j = 0$

$$V_1^0 = max(F_1^0, 0.5(V_2^1 + V_2^0)) = max(F_1^0, \frac{1}{1.01}(0.6F_2^1 + 0.4F_2^0)) = max(0.5, 0.4) = 0.5$$

- $j = 1$

$$V_1^1 = max(F_1^1, \frac{1}{1.01}(0.6V_2^2 + 0.4V_2^1)) = max(F_1^1, \frac{1}{1.01}(0.6F_2^2 + 0.4F_2^1)) = max(0, 0.0099) = 0.0099$$

For $t = 0$:

- $j = 0$

$$V_0^0 = max(F_0^0, \frac{1}{1.01}(0.6V_1^1 + 0.4V_1^0)) = max(0, 0.2039) = 0.2039$$

Finding the optimal exercise time

To find the exercise time we need for any (n, j) to compare F_n^j with V_n^j starting now from $n = 0$ and moving in future:

- In general $F_n \leq V_n^j$
- Starting from $n = 0$ you have that $F_n^j < V_n^j$ meaning that you are better off keeping the option than exercising it.
- Then at some n^* (and state j^*) you get $F_{n^*}^{j^*} = V_{n^*}^{j^*}$ meaning that the value of exercising at this point equals the value of keeping so this is the first time that you may exercise: Carpe diem! This is the optimal exercise time
- It may be that this equality happens again for larger values of n^* but the optimal exercise time and state will be the first one (n^*, j^*).

In our case

- $(0,0)$

$$F_0^0 = 0 < V_0^0 = 0.2039 \quad \text{Do not exercise}$$

- $(1,0)$

$$F_1^0 = 0.5 = V_0^1 = 0.5 \quad \text{Exercise}$$

- $(1,1)$

$$F_1^1 = 0 < V_1^1 = 0.0099 \quad \text{Do not Exercise}$$

- $(2,0)$

$$F_2^0 = 0 = V_2^0 = 0$$

- $(2,1)$

$$F_2^1 = 0.025 = V_2^1 = 0.025$$

- $(2,2)$

$$F_2^2 = 0.975 = V_2^1 = 0.975$$

You must exercise at time $n^* = 1$ if the underlying gets to state $j^* = 0$.