Financial Mathematics

Stock models: Continuous time models

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Continuous time models are very popular in finance, especially since they can often provide closed form solutions for various quantities of interest, e.g. option prices.

There are two main categories of continuous time models:

Models displaying continuous stock price paths (Brownian motion based models)

Models displaying discontinuous stock price paths (jump diffusion models)

The building block of the models: Brownian motion

Definition

Brownian motion is a stochastic process B_t on $\mathbb{R}$ with the following properties

(*i*) If $t_0 < t_1 < ... < t_n$ then the random variables $B_{t_0}, B_{t_1} - B_{t_0}, ..., B_{t_n} - B_{t_{n-1}}$ are independent (independent increments)

(*ii*) If $s, t \geq 0$, then

$$P(B_{s+t} - B_s \leq x) = \int_{-\infty}^{x} \frac{1}{(2\pi t)^{1/2}} exp\left(-\frac{\mid z \mid^2}{2t}\right) dz,$$

(*iii*) $t \to B_t$ is a continuous function with probability 1

Example

If B_t is a Brownian motion with $B_0 = 0$ then

$$E[f(B_t)] = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f(y) \exp\left(-\frac{y^2}{2t}\right) dy$$

Using that show $E[B_t^2] = t$.

Example (A model for stock prices)

A model for the price of a stock at time t, is that S_t follows

$$S_t = S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma B_t\right)$$

where $r, \sigma, S_0 > 0$ and B_t a Brownian motion.
What is the expectation $E[S_t]$;

Is S_t a martingale with respect to the filtration $\mathcal{F}_t$ generated by the Brownian motion?

We have

$$
\begin{aligned}
E[S_t] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} S_0 \exp\left((r - \frac{\sigma^2}{2})t + \sigma x\right) \exp\left(-\frac{x^2}{2t}\right) dx \\
&= \frac{S_0}{\sqrt{2\pi t}} \exp\left((r - \frac{\sigma^2}{2})t\right) \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2t} + \sigma x\right) dx = S_0 \exp(rt)
\end{aligned}
$$

This result shows that S_t is not a martingale, since for a martingale we would expect $E[S_t] = E[S_0]$ (not true since $\exp(r\,t) \neq 1$).

Pathological properties of Brownian motion

Theorem

- *$t \to B_t$ is nowhere differentiable with probability 1*
- *$t \to B_t$ is a function of infinite variation in any interval $[0, T]$*
- *$t \to B_t$ is a function of finite quadratic variation; Its quadratic variation in any interval $[0, T]$ is equal to T.*

Integrating over Brownian motion: The Itō integral

How can we define $\int_a^b f(t,\omega)dB_t(\omega)$?

Definition

Consider the partition $a = t_0 < t_1 < ... < t_n = b$ of $[a, b]$ and approximate the function $f(t,\omega)$ as

$$f(t,\omega) \simeq \sum_{i=0}^{n-1} f(t_i,\omega)\mathbf{1}_{[t_i,t_{i+1})}(t),$$

The Itō integral can be defined as the L^2-limit

$$\int_a^b f(t,\omega)dB_t(\omega) = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i,\omega)[B_{t_{i+1}} - B_{t_i}](\omega)$$

Note that

- The value of f at t must depend on B_s for $s \leq t$ **but not for** $s \geq t$ (i.e. f adapted!)
- The limit is considered in the L^2 sense and not for every ω (pointwise).

In the same way we may define the integral $\int_0^t f(s,\omega)dB_s(\omega)$ for every t and hence the stochastic process $\{M_t \; : \; t \in [0, T]\}$ where $M_t = \int_0^t f(s,\omega)dB_s(\omega)$.

$\{M_t, \; : \; t \in [0, T]\}$ is a martingale!

Properties of the Itō integral

Theorem

(1) *Linearity: For any* f_1, f_2 *and* $\lambda_1, \lambda_2 \in \mathbb{R}$

$$I(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 I(f_1) + \lambda_2 I(f_2)$$

(2) $E\left[\int_a^b f dB_t\right] = 0$

(3) $E\left[\left|\int_a^b f(t,\omega) dB_t\right|^2\right] = E\left[\int_a^b | f(t,\omega) |^2 \, dt\right]$

In all the above we assume that f *belongs to the right function space* M^2.

Itō processes

The Itō integral can be used to define a more general class of stochastic processes called Itō processes

Definition

A process X_t is called an Itō process if it is of the form

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s$$

where u and v satisfy

$$\int_0^t v^2(s,\omega)ds < \infty \ \ \sigma.\beta., \qquad \int_0^t u(s,\omega)ds < \infty \ \ \sigma.\beta.$$

In differential form

$$dX_t = udt + vdB_t$$

Itō's lemma

Theorem

Let X_t be an Itō process

$$X_t = X_0 + \int_0^t u(s,\omega)ds + \int_0^t v(s,\omega)dB_s.$$

Then given any function $(t,x) \mapsto g(t,x)$ in $C^{1,2}$:

$$g(t,X_t) = g(0,X_0) + \int_0^t \left(\frac{\partial g}{\partial s} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right) ds + \int_0^t v\frac{\partial g}{\partial x} dB_s$$

or in differential form

$$dg(t,X_t) = \left(\frac{\partial g}{\partial t} + u\frac{\partial g}{\partial x} + \frac{1}{2}v^2\frac{\partial^2 g}{\partial x^2}\right) dt + v\frac{\partial g}{\partial x} dB_t$$

Geometric Brownian motion

The simplest model with a single risk factor

Between t and $t+h$ the change in stock price is given by

$$\frac{S(t+h)-S(t)}{S(t)} = \mu\, h + \sigma(W(t+h)-W(t)), \quad t \geq 0$$
$$S(0) = S_0,$$

where

μ is related to the mean return

$\sigma > 0$ is the volatility

An equivalent form is

$$\frac{S(t+h)-S(t)}{S(t)} = \mu\, h + \sigma \sqrt{h} Z, \quad , Z \sim N(0,1), \;\; t \geq 0$$
$$S(0) = S_0,$$

In the limit $h \to 0$, the model yields the stochastic differential equation (SDE)

$$dS(t) = \mu S(t) dt + \sigma\, S(t) dW(t),$$
$$S(0) = S_0,$$

or the Itō integral

$$S(t) = S_0 + \int_0^t \mu S(r) dr + \int_0^t \sigma S(r) dW(r).$$

Define $Y(t) := \ln(S(t))$ and use Itō 's lemma to show

$$dY(t) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW(t),$$

Integrate and exponentiate:

$$S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t)\right)$$

We get the lognormal distribution (recall the binomial model)

$$\ln\left(\frac{S(t)}{S(0)}\right) \sim N\left(\left(\mu - \frac{\sigma^2}{2}\right) t, \sigma^2 t\right)$$

With a little effort (exercise)

$$\mathbb{E}[S(t)] = S(0)\exp\left(\mu t\right),$$
$$Var(S(t)) = S(0)^2 \exp\left(2\mu t\right)\,\left(\exp(\sigma^2 t) - 1\right).$$

To simulate GBM consider the partition $t_1 < t_2 < \cdots, < t_n$, and use

$$S(t_{i+1}) = S(t_i)\,\exp\left(\left(\mu - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i} Z_{i+1}\right),$$

where Z_i iid $Z_i \sim N(0,1)$.

Checking independence of returns

Calculate the autocorrelation function

$$C(k) = \frac{1}{(n-k)\hat{v}} \sum_{i=1}^{n-k} (x_i - \hat{\mu})(x_{i+k} - \hat{\mu}), \quad k = 1, 2, \cdots$$

and check decay with respect to k.

$\hat{\mu}$, $\hat{v}$ are the sample mean and variance of the time series of returns.

Estimators for μ and σ

Maximum likelihood estimator – using independence of returns

$$X(t_i) = \ln S(t_i) - \ln S(t_{i-1}) \sim N(a(t_{i+1} - t_i), \sigma^2(t_{i+1} - t_i)), \;\; a = \mu - \frac{1}{2}\sigma^2.$$

The likelihood of returns $x_1, \cdots, x_n$ is of the form

$$\mathcal{L}(\mu, \sigma^2) = \prod_{i=1}^{n} f(x_i; \mu, \sigma^2),$$

$$f(x_i; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2(t_{i+1} - t_i)}} \exp\left(-\frac{(x_i - a(t_{i+1} - t_i))^2}{2\sigma^2(t_{i+1} - t_i)}\right)$$

The estimators μ and σ^2 are then

$$m = \left(\hat{\mu} - \frac{1}{2}\hat{\sigma^2}\right)\Delta t,$$
$$v = \hat{\sigma^2}\Delta t, \ \ \Delta t = t_{i+1} - t_i,$$

where

$$m = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
$$v = \frac{1}{n}\sum_{i=1}^{n}(x_i - m)^2.$$

Ornstein-Uhlenbeck process

The Ornstein-Uhlenbeck process is an Itō process displaying returns to the mean.
This is a good model for modelling commodity prices, bond yields etc.

It solves the SDE

$$dX(t) = \theta(\nu - X(t))dt + \sigma dW(t), \;\; \theta > 0, \nu \in \mathbb{R}.$$

Using Ito's lemma

$$X(t) = e^{-\theta t} X_0 + \nu(1 - e^{-\theta t}) + \sigma e^{-\theta t} \int_0^t e^{\theta s} dW(s).$$

It is easy to see that

$$X(t) = e^{-\theta(t-s)} X_s + \nu(1 - e^{-\theta(t-s)}) + \sigma e^{-\theta t} \int_s^t e^{\theta u} dW(u),$$

leading to the discretized form

$$x(t_i) = c + b\, x(t_{i-1}) + \delta \epsilon(t_i),$$

where

$$c = \nu(1 - e^{-\theta \Delta t}), \;\; b = e^{-\theta \Delta t},$$
$$\delta = \sigma \sqrt{(1 - e^{-2\theta \Delta t})/2\theta}.$$

The Ornstein-Uhlenbeck is a Gaussian process with

$$\mathbb{E}[X(t)] = e^{-\theta t}\mathbb{E}[X_0] + \nu(1 - e^{-\theta t}),$$
$$Cov(X(s), X(t)) = \frac{\sigma^2}{2\theta} e^{-\theta(s+t)} \left(e^{2\theta min(s,t)} - 1 \right),$$
$$Var(X(t)) = \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}).$$

For $t \to \infty$, $X(t) \to Y$ in distribution with $Y \sim N\left(\nu, \frac{\sigma^2}{2\theta}\right)$

Moreover,

$$E_s[X(t)] = \nu + (x(s) - \nu)e^{-\theta(t-s)},$$
$$Var_s[X(t)] = \frac{\sigma^2}{2\theta}(1 - e^{-\theta(t-s)}).$$

The following estimators can be used for calibration

$$\hat{b} = \frac{n\sum_{i=1}^{n} x(i)x(i-1) - \sum_{i=1}^{n}\sum_{i=1}^{n} x(i-1)}{n\sum_{i=1}^{n} x(i-1)^2 - (\sum_{i=1}^{n} x(i-1))^2},$$

$$\hat{\nu} = \frac{\sum_{i=1}^{n}(x(i) - \hat{b}x(i-1)}{n(1-\hat{b})},$$

$$\hat{\delta}^2 = \frac{1}{n}\sum_{i=1}^{n}(x(i) - \hat{b}x(i-1) - \hat{\nu}(1-\hat{b}))^2.$$

Connection with $AR(1)$ models

The solution of the OU process

$$X(t) = e^{-tA}X(0) + \int_0^t e^{-(t-s)A} S dW(s),$$

if observed at discrete times $t_i = ih$ yields the $AR(1)$ model

$$Y_i = e^{-hA} Y_{i-1} + \xi_i,$$
$$\xi_i = \int_{i-1}^{i} e^{-(kh-s)A} S dW(s), \quad i = 0, 1, \cdots$$

The equivalent martingale measure

Definition

A market model is a $\mathcal{F}_t = \sigma(B(s) = (B_1(s), \cdots, B_d(s)) \; : \; s \in [0, t])$ adapted Itō process $X(t) = (X_0(t), X_1(t), \cdots, X_n(t))$, $0 \leq t \leq T$, of the form

$$dX_0(t) = r\, X_0(t), \;\; X_0(0) = 1,$$
$$dX_i(t) = \mu_i dt + \sum_{j=1}^{d} \sigma_{ij} dB_j, \;\; i = 1, \cdots, d.$$

Definition

A portfolio in the market $\{X(t)\}_{t\in[0,T]}$ is a $n+1$-dim $\mathcal{F}_t$ adapted process

$$\theta(t) = (\theta_0(t), \theta_1(t), \cdots, \theta_n(t)), \quad t \in [0, T]$$

where $\theta_i(t)$ is the position of the investor in asset i at t.

The value of the portfolio θ at t is

$$V(t) = \sum_{i=0}^{n} \theta_i(t) X_i(t).$$

Definition

A portfolio is called self-financing if

$$V(t) = V(0) + \int_0^t \theta(s) dX(s), \quad t \in [0, T].$$

In terms of the discounted processs

$$X^*(t) = \frac{1}{X_0(t)} X(t),$$

$$V^*(t) = \frac{1}{X_0(t)} X(t)$$

we can show

$$dV^*(t) = \sum_{j=1}^{n} \theta_j(t) dX_j^*(t).$$

Equivalent measures

Definition

The measure P is absolutely continuous with respect to Q, denote $P << Q$, if for every $A \subset \Omega$ such that $Q(A) = 0$ we have $P(A) = 0$.

Definition

P and Q are equivalent, denote $P \sim Q$, if

$$P << Q \text{ and } Q << P.$$

Two equivalent measures have the same null sets

Theorem (Radon-Nikodym)

If $P << Q$ there exists a unique $Z \in L^1(\Omega, \mathcal{F}, Q)$ such that

$$P(A) = \int_A Z dQ, \ \ \forall A \in \mathcal{F}.$$

Z is called the Radon-Nikodym derivative of P with respect to Q, denote $Z = \frac{dP}{dQ}$.

If $P << Q$ and $Z = \frac{dP}{dQ}$ then for any integrable r.v.

$$\int_\Omega X dP = \int_\Omega X \frac{dP}{dQ} dQ,$$

therefore,

$$\mathbb{E}_P[X] = \mathbb{E}_Q[XZ] = \mathbb{E}_Q\left[X \frac{dP}{dQ}\right]$$

Definition

Q is called an equivalent martingale measure if $Q \sim P$ and

$$\mathbb{E}_Q[X_j^*(t) \mid \mathcal{F}_s] = X_j^*(s), \;\; j = 1, \cdots, n.$$

If a portfolio is self-financing $V^*(t)$ is a Q-martingale

$$\mathbb{E}_Q[V^*(t) \mid \mathcal{F}_s] = V^*(s).$$

Definition

An arbitrage opportunity is a self-financing portfolio θ such that $V(T) \geq 0$ a.s. and $P(V(T) > 0) > 0$.

Theorem

There are no arbitrage opportunities for a market model if and only if there exists and equivalent martingale measure

Girsanov's theorem

Theorem

Let $Y(t) \in \mathbb{R}^d$ be the Ito process

$$dY(t) = a(t,\omega)dt + dB(t), \quad Y(0) = 0, \quad t \leq T,$$

where $B(t) = (B_1(t), \cdots, B_d(t))$ is a d-dimensional Brownian motion and

$$M(t) = \exp\left(-\int_0^t a(s,\omega)dB(s) - \frac{1}{2}\int_0^t a^2(s,\omega)ds\right).$$

Assume the Novikov condition

$$\mathbb{E}_P\left[\exp\left(\frac{1}{2}\int_0^t a^2(s,\omega)ds\right)\right] < \infty$$

and define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$\frac{dQ}{dP} = M(T).$$

Then $Y(t)$ is a d-dim Brownian motion with respect to the measure Q, for $t \leq T$.

There exists a measure change from P to$Q$ such that if under P the process $Y(t)$ is

$$dY = \beta dt + \theta dB,$$

then under Q, Y is the process

$$dY = \alpha dt + \theta d\bar{B}.$$

This is immediate as long as we find a measure Q under which

$$\bar{B}(t) = \int_0^t u(s,\omega)ds + B(t), \;\; t \leq T,$$

is a Brownian motion, for u which solves the linear equation

$$\beta(t,\omega) - \theta(t,\omega)u(t,\omega) = \alpha(t,\omega).$$

The above observation can help us show the existence of an equivalent martingale measure

If under P

$$dX_i(t) = \mu_i dt + \sum_{k=1}^{d} \sigma_{ik} dB_k(t), \quad i = 1, \cdots, n$$

then to get to an EMM we need to get to a measure Q such that under it

$$dX_i(t) = rX_i dt + \sum_{k=1}^{d} \sigma_{ik} dB_k(t), \quad i = 1, \cdots, n$$

The linear equation

$$\beta(t,\omega) - \theta(t,\omega)u(t,\omega) = \alpha(t,\omega).$$

will allow us to find the process u for the measure change.

Exercise: Do it for the GBM!