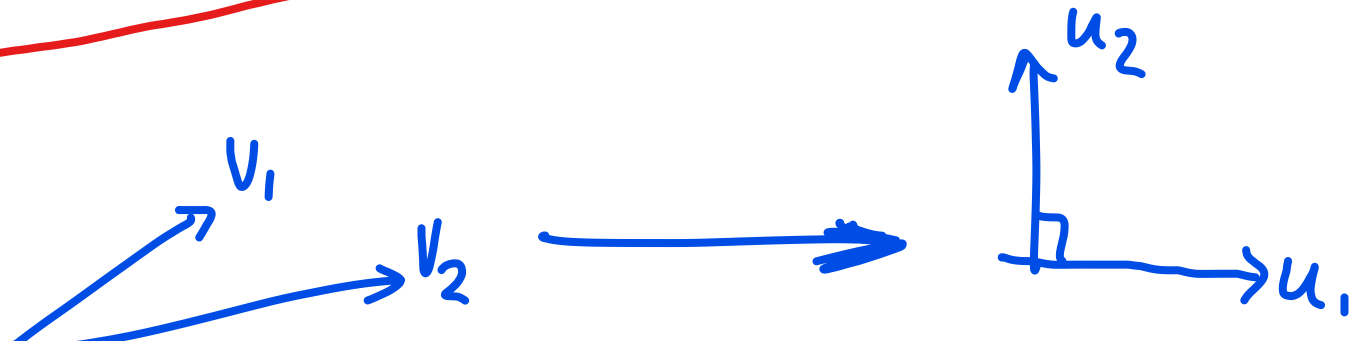
16 Μαρτίου :

$\boxed{\text{Gram-Schmidt}}$

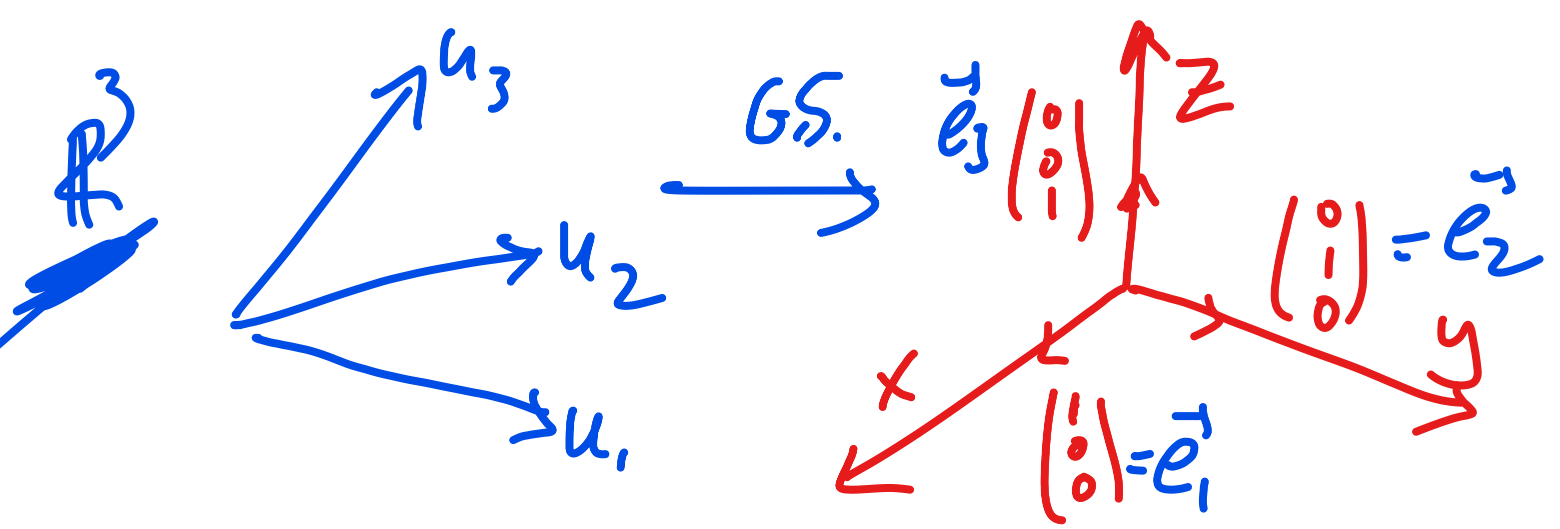
$\boxed{\text{Ορθοκανονικοποίηση G-S , QR}}$



Γ.Α. $\vec{v}_1, \vec{v}_2$ $\qquad$ $\|u_i\| = 1$

$\{v_i\}, \{u_i\}$ = βάσεις του ίδιου χώρου

$\boxed{\text{Βάση } v_i \xrightarrow{\text{G.S.}} \text{Βάση } u_i = \text{ΟΚΒ}}$



Ⓐ Θέτω $\vec{u}_1 = \vec{v}_1$.

$\vec{v}_i$ = γνωστά

Ⓑ $\vec{u}_2 = \vec{v}_2 - \dfrac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot \vec{u}_1$

Αφαιρούμε τα μη κάθετα "μέρη"

Ⓖ $\vec{u}_3 = \vec{v}_3 - \dfrac{\langle v_3, u_1 \rangle}{\|u_1\|^2} \vec{u}_1 - \dfrac{\langle v_3, u_2 \rangle}{\|u_2\|^2} \vec{u}_2$

ΚΛΠ. ΟΜΟΙΩΣ

Ⓓ ΚΑΝΟΝΙΚΟΠΟΙΗΣΗ ΤΩΝ $\vec{u}_i$

Πχ 1: Έστω $\mathbb{R}^4$: $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix}$

$\vec{v}_3 = \begin{pmatrix} 0 \\ 3 \\ 6 \\ 5 \end{pmatrix}$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$: Γ.Α.

$\lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3 = 0 \Rightarrow \lambda_i = 0$

$[v_1, v_2, v_3] =$ ς χώρος $\subseteq \mathbb{R}^4 = [u_1, u_2, u_3]$?

G.S.

(A) Θέτω $\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$

(B) $$\vec{u}_2 = \vec{V}_2 - \frac{\langle v_2, u_1 \rangle}{\|u_1\|^2} \cdot \vec{u}_1 =$$

$$\begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix} - \boxed{\frac{\left\langle \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle}{\sqrt{1^2+0^2+1^2+0^2}^{\,2}}} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

αριθμός

$$= \begin{pmatrix} 2 \\ 2 \\ 0 \\ -3 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ -3 \end{pmatrix}$$

$$= \vec{u}_2 .$$

r $\vec{u}_3 = v_3 - \dfrac{\langle v_3, u_1\rangle}{\|u_1\|^2}\cdot \vec{u}_1 - \dfrac{\langle v_3, u_2\rangle}{\|u_2\|^2}\cdot \vec{u}_2$

$$= \begin{pmatrix}0\\3\\6\\5\end{pmatrix} - \frac{\begin{pmatrix}0\\3\\6\\5\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix}}{\sqrt{1^2+1^2}^{\,2}}\cdot \vec{u}_1 - \frac{\begin{pmatrix}0\\3\\6\\5\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-1\\-3\end{pmatrix}}{\sqrt{1+9+4+1}^{\,2}}\cdot \vec{u}_2$$

$$\begin{pmatrix}0\\3\\6\\5\end{pmatrix} - \frac{6}{2}\cdot\begin{pmatrix}1\\0\\1\\0\end{pmatrix} - \frac{-15}{15}\begin{pmatrix}1\\2\\-1\\-3\end{pmatrix}$$

$$= \begin{pmatrix}0\\3\\6\\5\end{pmatrix} - \begin{pmatrix}3\\0\\3\\0\end{pmatrix} + \begin{pmatrix}1\\2\\-1\\-3\end{pmatrix} = \underbrace{\begin{pmatrix}-2\\5\\2\\2\end{pmatrix}}_{\vec{u}_3}$$

$$\vec{u}_1 = \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix}1\\2\\-1\\-3\end{pmatrix}, \quad \vec{u}_3 = \begin{pmatrix}-2\\5\\2\\2\end{pmatrix}$$

Δ Κανονικοποιώ:

$$\begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{15} \\ 2/\sqrt{15} \\ -1/\sqrt{15} \\ -3/\sqrt{15} \end{pmatrix}$$

$$\begin{pmatrix} -2/\sqrt{37} \\ 5/\sqrt{37} \\ 2/\sqrt{37} \\ 2/\sqrt{37} \end{pmatrix}$$

OK.

$n \times 2$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

~~G-S~~ ?

A $\vec{u}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\boxed{\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 11}$$

B $\vec{u}_2 = \vec{v}_2 - \dfrac{\langle u_1, v_2 \rangle}{\|u_1\|^2} \cdot \vec{u}_1 =$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{11}{5} \begin{pmatrix} 1 \\ 2 \end{pmatrix} =$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 11/5 \\ 22/5 \end{pmatrix} = \begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix}$$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \vec{u}_2 = \begin{pmatrix} 4/5 \\ -2/5 \end{pmatrix}$$

$$\boxed{\|u_2\| = \sqrt{\frac{20}{25}} = \frac{\sqrt{20}}{5}}$$

r $\quad \vec{u}_1 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$

$\vec{u}_2 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

OKB

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$

v_1, v_2

Η ΠΑΡΑΓΟΝΤΟΠΟΙΗΣΗ QR

A: πλήρους τάξης

full rank

$$A = Q \cdot R$$

Q: $m \times n$ ορθογώνιος, $Q^T Q = I$

A: $m \times n$

R: $n \times n$ Άνω τριγωνικός αντιστρέψιμος

$$\boxed{24 = 6 \cdot 4}$$

$A = LDU = LDL^T =$

Cholesky: $LD^{1/2} D^{1/2} L^T = R \cdot R^T$

$\boxed{A = A^T}$

ΠΑΡΑΓΟΝΤΟΠΟΙΗΣΕΙΣ:

$A = LU$	$A = QR$
Απαλοιφή Gauss	Κανονικοποίηση G-S

A: n×n, αντιστρέψ. (full rank) | A = m×n, full rank

(ΜΟΝΑΔΙΚΕΣ)

QR

A ⟶ G.S. (στήλες)

⟶ νέες στήλες = Q ορθογώνιος

$$A = QR \Rightarrow Q^T A = Q^T Q R \Rightarrow$$

$$\underline{R = Q^T A.}$$

Πχ: $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ QR?

$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \xrightarrow{G.S.} (n \times 2)$ $\begin{pmatrix} 1/\sqrt{5} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} 2/\sqrt{5} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

$$Q = \begin{bmatrix} 1/\sqrt{5} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix}$$

~~O R?~~ $\quad R = Q^T A =$

$$\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \overset{R}{\begin{bmatrix} \sqrt{5} & 11/\sqrt{5} \\ 0 & \frac{2}{\sqrt{5}} \end{bmatrix}}$$

(R: θετική διαγώνιος)

Ιδιότητες - Χρησιμότητα:

$$A = Q \cdot R$$

1) Σύστημα $Ax=b \Rightarrow$

$QRx=b \Rightarrow \underline{Rx=Q^Tb}$

Τετραγωνικό σύστημα $\rightarrow$ $\underline{\text{γρήγορο}}$

2) $\boxed{\underline{\text{Λ.Ε.Τ.}}\text{ της } Ax=b \Leftrightarrow A^TAx=A^Tb}$ $\Longrightarrow$ ίδια λύση με την $Rx=Q^Tb$

$\underline{\text{Αποδ.}}$: Έστω $A=QR \rightarrow A^T=R^TQ^T$.

Κανονική Εξίσωση: $\boxed{A^TAx=A^Tb} \Rightarrow R^T\underbrace{Q^TQ}Rx=R^TQ^Tb$

$\Longrightarrow R^T\cdot Rx=R^T\cdot Q^T\cdot b \quad \cdot(R^T)^{-1}$

$\Rightarrow \boxed{Rx=Q^Tb}$

Στήλες του A: γ.Α.

$\Updownarrow$

Αν ο $A=$ full rank