19 Μαΐου | Φροντιστήριο |

SVD_Polar

① Πολική διάσπαση $A = \begin{bmatrix} 1{,}3 & -0{,}375 \\ 0{,}75 & 0{,}65 \end{bmatrix}$

$A = Q|A|$

$|A| = (A^T A)^{1/2}$ ρίζα

$\boxed{\text{Θεωρία}}$

Q = ορθογώνιος, $Q^T = Q^{-1}$
(στροφή)

και: $Q = (A^T)^{-1} \cdot |A|$

Πρώτα: $A^T A = \begin{bmatrix} 1{,}3 & 0{,}75 \\ -0{,}375 & 0{,}65 \end{bmatrix} \begin{bmatrix} 1{,}3 & -0{,}375 \\ 0{,}75 & 0{,}65 \end{bmatrix}$

=

$$\begin{bmatrix} 2,25 & 0 \\ 0 & 0,563 \end{bmatrix} = T$$

$$|A| = T^{1/2} = \begin{bmatrix} \sqrt{2,25} & 0 \\ 0 & \sqrt{0,563} \end{bmatrix} = \begin{bmatrix} 1,5 & 0 \\ 0 & 0,751 \end{bmatrix}$$

① Q: $(A^T)^{-1} \cdot |A| = \begin{bmatrix} 1,3 & 0,75 \\ -0,375 & 0,65 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1,5 & 0 \\ 0 & 0,75 \end{bmatrix}$

$$= \begin{bmatrix} 0,577 & -0,666 \\ 0,333 & 1,15 \end{bmatrix} \cdot \begin{bmatrix} 1,5 & 0 \\ 0 & 0,75 \end{bmatrix} =$$

$$\begin{bmatrix} 0,866 & -0,5 \\ 0,5 & 0,866 \end{bmatrix} = Q$$

ορθογώνιος $(Q^{-1} = Q^T)$ $\quad \mathbb{R}^{2\times 2} =$ $\boxed{\text{στροφής}}$

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \circlearrowleft \theta$$

Εδώ $\theta = \frac{\pi}{6}$ $\circlearrowleft$

$$A = \begin{bmatrix} 0{,}866 & -0{,}5 \\ 0{,}5 & 0{,}866 \end{bmatrix} \cdot \begin{bmatrix} 1{,}5 & 0 \\ 0 & 0{,}751 \end{bmatrix}$$

στροφή 30° $\circlearrowleft$

πληροφορία

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② SVD : «Διαγωνοποίηση»

$$A = U \Sigma V^T$$

ορθογώνιοι

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ -1 & -2 & 1 & 3 \end{bmatrix}$$

(2×4)

$$\boxed{\mathrm{rank}(A) = 2}$$

$\Sigma =$

rank 2

U:

$$AA^T = \begin{bmatrix} 7 & -3 \\ -3 & 15 \end{bmatrix}$$

$(2\times4)\cdot(4\times2)$

<u>Ιδιοτιμές</u>:

$$\boxed{\begin{matrix} \lambda_1 = 16 \\ \lambda_2 = 6 \end{matrix}} \begin{matrix} \to \sigma_1 = 4 \\ \to \sigma_2 = \sqrt{6} \end{matrix}$$

$$\vec{u} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad (\vec{u} \perp \vec{v})$$

<u>ΚΑΝΟΝΙΚΟΠΟΙΗΣΗ</u>:

$$\vec{u} = \begin{pmatrix} -1/\sqrt{10} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$U = \begin{bmatrix} -1/\sqrt{10} & \frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix}$$

Ο Σ : 2×4, $\text{rank}(\Sigma) = 2$, $\sigma_1 = 4$, $\sigma_2 = \sqrt{6}$

$$\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \sqrt{6} & 0 & 0 \end{bmatrix}$$

Ο V : από τον $\underset{4\times 2}{A^T}\,\underset{2\times 4}{A} = 4\times 4$

$$= \begin{bmatrix} 2 & 4 & -2 & -2 \\ 4 & 8 & -4 & -4 \\ -2 & -4 & 2 & 2 \\ -2 & -4 & 2 & 10 \end{bmatrix}$$

τάξη $= 2$

4 ιδιοτιμές : $16, 6, 0, 0 \rightarrow$ διπλή ιδιοτιμή

Ιδιοδιανύσματα :

$$\vec{x}_1 = \begin{pmatrix} -1 \\ -2 \\ 1 \\ 2 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 3 \end{pmatrix} \quad \vec{x}_3 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{x}_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Θα κάνω Gram-Schmidt στο $\vec{x}_4$ $\qquad$ $\vec{x}_4$ όχι $\perp$ $\vec{x}_3$

$$\hat{x}_4 = x_4 - \frac{\langle x_3, x_4 \rangle}{\|x_3\|^2}\vec{x}_3 =$$

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} - \frac{-2}{5}\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{array}{c} \vec{x}_4 \\ \begin{pmatrix} 1/5 \\ 2/5 \\ 1 \\ 0 \end{pmatrix} \end{array}$$

Κανονικοποίηση στα:

$\vec{x}_1, \vec{x}_2, \vec{x}_3, \hat{x}_4$

Βάζω σε στήλες στον V:

$$
V = \begin{bmatrix} -1/\sqrt{10} & 1/\sqrt{15} & -\frac{2}{\sqrt{15}} & \frac{1}{\sqrt{30}} \\ -2/\sqrt{10} & 2/\sqrt{15} & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} \\ 1/\sqrt{10} & -\frac{1}{\sqrt{15}} & 0 & \frac{5}{\sqrt{30}} \\ 2/\sqrt{10} & \frac{3}{\sqrt{15}} & 6 & 0 \end{bmatrix}
$$

$$
A = U\Sigma V^T
$$

Σύνδεση SVD – Polar

$$
A = Q|A| = \underbrace{(UV)}_{Q} \cdot \underbrace{(V\Sigma V^T)}_{|A|}
$$

Αν
$A = U\Sigma V^T$

3

Πολική: $A = \begin{bmatrix} \frac{10}{\sqrt{10}} & \frac{6}{\sqrt{10}} \\ 0 & \frac{8}{\sqrt{10}} \end{bmatrix}$

Φασματικό θεώρημα:

$$A^TA = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

ρίζα?

$$|A| = (A^TA)^{1/2}$$

Ιδιοτιμές: $4, 16$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Διαγωνοποίηση.

$$A^TA = PD \cdot P^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Λσ. $(A^TA)^{1/2} = \begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

πράξεις $\Rightarrow (A^TA)^{1/2} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = |A|$

$\boxed{A = Q \cdot |A|}$ ή $\boxed{A = |A| . U}$

$\to$ ή $Q = A \cdot |A|^{-1} = \begin{bmatrix} \frac{10}{\sqrt{10}} & 6/\sqrt{10} \\ 0 & \frac{8}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$

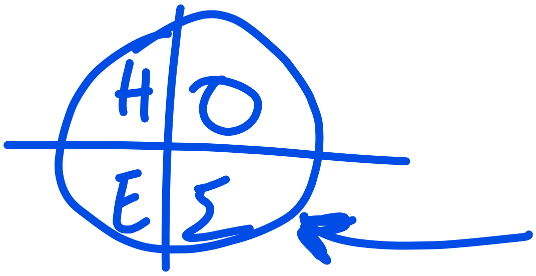
$Q = (A^T)^{-1} \cdot |A|$

$= \dots \underbrace{\begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{-1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}}_{Q}$

πίνακας στροφής

$\cos\theta = \frac{3}{\sqrt{10}}$

$\sin\theta = -\frac{1}{\sqrt{10}}$



$18{,}5^\circ$ στροφή

$$A = Q \cdot |A|$$

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4) $A = \begin{bmatrix} 3 & -3 \\ -3 & -5 \end{bmatrix}$ $\quad \lambda_1 = -6$, $\lambda_2 = k$

α) k?

β) A^{10}?

α) $\boxed{\mathrm{trace}(A) = \sum \lambda_i}$

$$\Rightarrow 3 - 5 = -6 + k \Rightarrow \boxed{k = 4}$$

β) Ιδιοδιανύσματα

$A\vec{u} = -6\vec{u} \to \ldots$

$\vec{u} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, $A\vec{v} = 4\vec{v} \to \ldots$ $\vec{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$

Διαγωνοποίηση:

$$A = P \cdot D \cdot P^{-1} =$$

$$\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -6 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \frac{1}{10} & \frac{3}{10} \\ -\frac{3}{10} & \frac{1}{10} \end{bmatrix} \Rightarrow$$

$$A^{10} = P \cdot \begin{bmatrix} (-6)^{10} & 0 \\ 0 & 4^{10} \end{bmatrix} P^{-1} \ldots$$

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5 $A = \begin{bmatrix} 0,4 & -0,3 \\ 0,4 & 1,2 \end{bmatrix}$

α) διαγωνοποιείται?

β) $\lim_{k \to \infty} A^k = ?$

Ιδιοτιμές:

$$\begin{vmatrix} 0,4-\lambda & -0,3 \\ 0,4 & 1,2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (0,4-\lambda)(1,2-\lambda)+0,12=0 \Rightarrow$$

$$\lambda=1 \;,\; \lambda=0,6$$

<u>Διανύσματα</u>: $\vec{u}=\begin{pmatrix}-1\\2\end{pmatrix}$ $\quad \vec{v}=\begin{pmatrix}-3\\2\end{pmatrix}$

<u>ΝΑΙ</u>.

$$A=\begin{bmatrix}-1 & -3\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0,6\end{bmatrix}\begin{bmatrix}-1 & -3\\ 2 & 2\end{bmatrix}^{-1}$$

β) $$A^{k}=\begin{bmatrix}-1 & -3\\ 2 & 2\end{bmatrix}\begin{bmatrix}1^{k} & 0\\ 0 & 0,6^{k}\end{bmatrix}\cdot\begin{bmatrix}\frac{2}{4} & \frac{3}{4}\\ \frac{-2}{4} & \frac{-1}{4}\end{bmatrix}$$

Άρα $$\lim_{k\to\infty} P\begin{bmatrix}1^{k} & 0\\ 0 & 0,6^{k}\end{bmatrix}P^{-1}=$$

$$P\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}P^{-1}=$$

$$\begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{2}{4} & \frac{3}{4} \\ -\frac{2}{4} & -\frac{1}{4} \end{bmatrix} =$$

$$A^K = \begin{bmatrix} -1/2 & -3/4 \\ 1 & 3/2 \end{bmatrix}$$

$K \to +\infty$