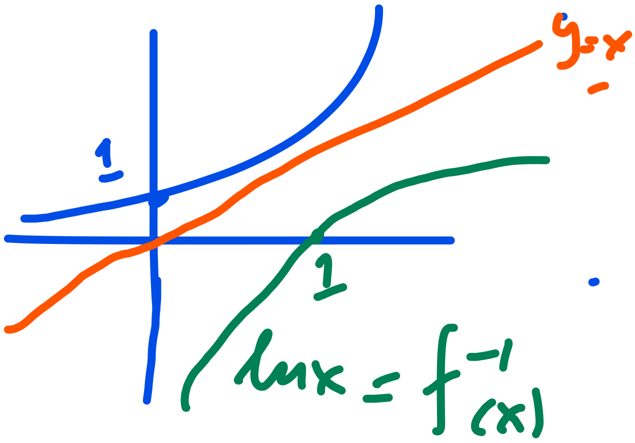
22 Απρ.

$$f(x) = e^{kx}$$



$$f(x) \longrightarrow e^{Ax} \qquad \boxed{A = n \times n}$$

= πίνακας

McLaurin

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots}$$

Θέμα!:

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

e^{At} = πίνακας | ΕΚΘΕΤΙΚΟΣ ΠΙΝΑΚΑΣ

στηρίζεται στο Mc Laurin.

(e^{At}, $\sin A$, $\cos A$ $\longrightarrow$ $A^{n\times n}$)

(A) ① A διαγωνοποιείται:

λ_i, $\vec{u}_i$ $\longrightarrow$ $\Delta = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}$

$$\boxed{A = P\Delta P^{-1}}$$

$$e^{At} = Pe^{\Delta t}P^{-1} = P\begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots \; e^{\lambda_n t} \end{bmatrix}P^{-1}$$

πχ1: $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$

$$(2+\lambda)^2 - 1 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -3 \end{cases}$$

<u>Ιδιοδιανύσματα</u> $\vec{u}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \vec{u}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \left(Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \left(-\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \right)$$

Άρα $e^{At} = P e^{\Delta t} P^{-1} =$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}}_{} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \dots \begin{bmatrix} \frac{e^{-t}+e^{-3t}}{2} & \frac{e^{-t}-e^{-3t}}{2} \\ \frac{e^{-t}-e^{-3t}}{2} & \frac{e^{-t}+e^{-3t}}{2} \end{bmatrix}$$

$$= e^{At}$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

B Αν ο A δεν διαγωνοποιείται

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\boxed{x \longrightarrow At}$$ γίνεται:

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

(συγκλίνει πάντα)

Ιδιότητες: $e^{At} \cdot e^{As} = e^{A(t+s)}$

$e^{0} = I$, $e^{At} \cdot e^{-At} = I$

$\boxed{\text{Θέτω: } e^{At} = B}$ $\quad e^{At} \to B, \quad e^{-At} \to B^{-1}$

Άρα e^{At} ΠΑΝΤΑ αντιστρέφεται.

πχ2

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad e^{At}?$$

ΠΙΝΑΚΑΣ

Ιδιοτιμές ?

$$\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$$

$$\boxed{\lambda \notin \mathbb{R}} \Rightarrow \lambda = \pm i$$

McLaurin :

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$= I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \frac{A^4t^4}{24}$$

??

ή :

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} &A^2 + I = 0 \\ &\Rightarrow A^2 = -I \end{aligned}}$$

$$A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$$

MOTIBO

$$\begin{cases} A^2 = -I \\ A^3 = A\cdot A^2 = -A \\ A^4 = A^2\cdot A^2 = (-I)\cdot(-I) = I \\ A^5 = A^4\cdot A = A \quad \ldots\ldots \end{cases}$$

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \frac{A^4t^4}{24} + \ldots =$$

$$I + At - \frac{It^2}{2} - \frac{At^3}{6} + \frac{It^4}{24} + \ldots =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} t - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{t^2}{2} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \frac{t^3}{6} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \frac{t^4}{24}$$

$$= \begin{bmatrix} 1+0-\frac{t^2}{2}+\frac{t^4}{24} & -t+\frac{t^3}{3}\cdots \\ t-\frac{t^3}{6}+\cdots & 1-\frac{t^2}{2}+\frac{t^4}{24}\cdots \end{bmatrix}$$

(McLaurin)

$$= \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = e^{At},$$

$$\boxed{\text{KATI ΣAN: } \left(e^{i\theta} = \cos\theta + i\sin\theta \right)}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \Rightarrow$$

$x \longmapsto A^{n\times n}$

$$\sin A = A - \frac{1}{3!}A^3 + \frac{1}{5!}A^5 - \cdots$$

ΕΑΝ·

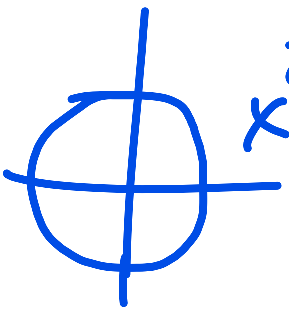
πχ <u>διαγωνοποιείται:</u>

$$\sin A = P \sin(\Delta) P^{-1} =$$

$$P \begin{bmatrix} \sin\lambda_1 & & & 0 \\ & \sin\lambda_2 & & \\ & & \ddots & \\ 0 & & & \sin\lambda_n \end{bmatrix} P^{-1} \ldots$$

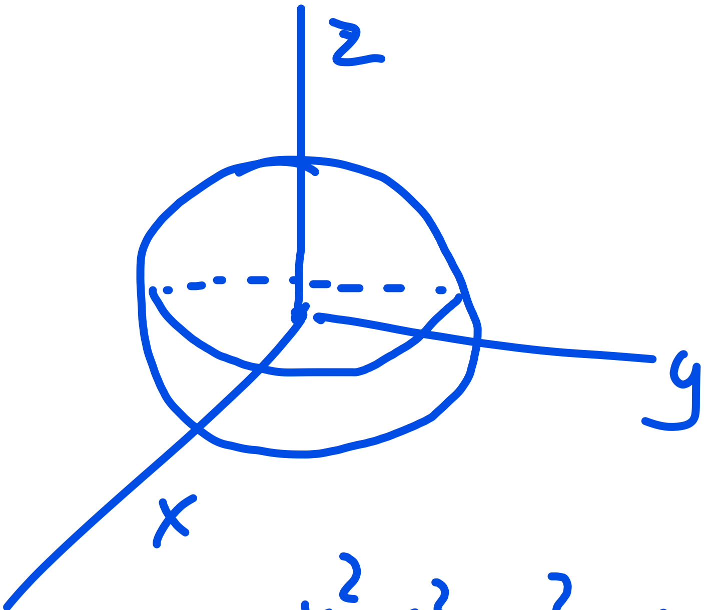
ΑΚΡΟΤΑΤΑ ΤΕΤΡΑΓΩΝΙΚΩΝ ΜΟΡΦΩΝ ΣΤΗΝ ΜΟΝΑΔΙΑΙΑ ΣΦΑΙΡΑ.

$\mathbb{R}^2$



$x^2+y^2=1$

Μοναδ. κύκλος

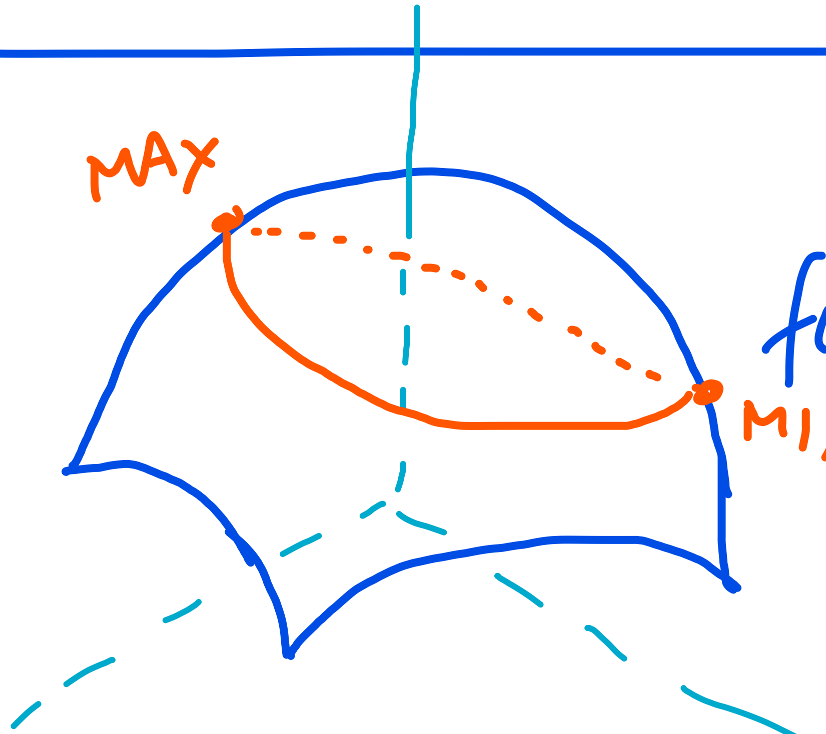


$x^2+y^2+z^2=1$

$f(x) =$ MIN? $x \cdot y$, με $x+y=8$

ΥΠΟ ΠΕΡΙΟΡΙΣΜΟΥΣ

Πχ πολ/στές Lagrange: Λογ. 2



$f(x,y) = x^T A x$

με $\|x\| = 1$

(Optimization)

$f(\vec{x}) = x^T A x$, (A συμμετρικός)

$\|x\| = 1$.

$$\begin{cases} f_{MAX} \longrightarrow \lambda_{MAX}. & \vec{x} = \text{ιδιοδιαν} \\ f_{MIN} \longrightarrow \lambda_{MIN} & \text{για } \vec{x} = \text{ιδιοδιαν.} \end{cases}$$

$\boxed{\text{παρ 1}}$ $f(x,y,z): \mathbb{R}^3 \longrightarrow \mathbb{R}$

$= 2xy + 2xz.$ $\quad \boxed{1 = \sqrt{x^2+y^2+z^2}}$

$f_{MAX}, f_{MIN}?$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Ιδιοτιμές: $|A-\lambda I| = 0 \Rightarrow$

$$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = \sqrt{2} \\ \lambda_2 = -\sqrt{2} \\ \lambda_3 = 0 \end{cases}$$

$f_{MAX} = \sqrt{2} \longrightarrow$ ιδιοδιάνυσμα $\vec{x}$: $\pm\begin{pmatrix} \frac{\sqrt{2}}{2} \\ 1/2 \\ 1/2 \end{pmatrix}$

$f_{MIN} = -\sqrt{2} \longrightarrow$ ιδιοδιάνυσμα $\vec{x}$: $\pm\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ 1/2 \\ 1/2 \end{pmatrix}$

$$\boxed{A\vec{u} = \lambda\vec{u}}$$

$\underline{\lambda = +\sqrt{2}}$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \sqrt{2}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \sqrt{2}x \\ \sqrt{2}y \\ \sqrt{2}z \end{bmatrix} \Rightarrow$$

$$\begin{cases} y + z = \sqrt{2}x \\ x = \sqrt{2}y \\ x = \sqrt{2}z \end{cases} \qquad (y = z)$$

Αν θέλω $\|\vec{x}\| = a$?

$$f_{MAX} = \lambda_{MAX} \cdot a \longrightarrow \text{ιδιοδιαν.}$$

$$f_{MIN} = \lambda_{MIN} \cdot a \longrightarrow \text{ιδιοδιαν.}$$

Πχ2:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$f(x) = x^T A x \longrightarrow f_{MAX}? \quad \left(\overset{\text{με}}{\|x\| = 1} \right)$$

$$\downarrow$$

$$\lambda_{MAX}$$

Αρα :

$$\begin{vmatrix} 3-\lambda & 2 & 1 \\ 2 & 3-\lambda & 1 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$-\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0$$

$$\lambda = 6, \quad \lambda = 3, \quad \lambda = 1$$

$$f_{max} = 6 = f(\vec{u})$$

$$A\vec{u} = 6\vec{u} \Rightarrow \ldots \qquad \vec{u} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \hat{u} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$