Stochastic Models in Insurance — Fall 2015-2016

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Problem 1. Let X_i, $i = 1, 2, \ldots$ be independent random variables exponentially distributed with mean m, i.e. $\mathbb{P}(X_i \leq x) = 1 - e^{-x/m}$, $x \geq 0$. Set $S_n := \sum_{i=1}^{n} X_i$. Suppose that $m = 10$.

1. What is the mean and variance of S_n? Use the Central Limit theorem in order to estimate the probability $\mathbb{P}(S_n > nx)$ for $x = 15, 20, 25, \ldots, 40$ when $n = 50$ and $n = 100$.
2. Estimate the same probabilities using simulation. Provide 95% confidence intervals for your estimates.
3. Compute the rate function $I(x) := \sup_\theta\{\theta x - \log M(\theta)\}$ where $M(\theta)$ is the moment generating function of X. Compute the Large Deviations estimate

$$\log \mathbb{P}(S_n > nx) \approx -nI(x)$$

and compare them with the other two approaches.

Problem 2. Consider the following Cramér–Lundberg risk model. Let $\{X_t; t \geq 0\}$ denote the free reserves process given by

$$X_t \;=\; u + ct - \sum_{i=1}^{N(t)} Z_i.$$

In the above expression u denotes the initial capital, c denotes the premium rate, $\{Z_i\}$ is a sequence of independent, identically distributed non negative random variables (with distribution $F(z)$) that represent the sizes of claims and finally $\{N(t); t \geq 0\}$ is a Poisson process with rate $\lambda > 0$, independent of the claim sizes $\{Z_i\}$. $N(t)$ is the number of claims that have occurred up to time t. Let $m := \mathbb{E}[Z_i]$ denote the mean size of each claim.

The premium rate is typically set at a value such that $c > \lambda m$ in order to insure that on the average the operation of the insurance company is profitable and therefore the likelihood of ruin is small. The *loading factor* $\rho > 0$ is defined via the relationship $\rho := \frac{c}{\lambda m} - 1$ and we assume that $\rho > 0$. We denote the *finite horizon ruin probability* as

$$\Psi(u, T) := \mathbb{P}(X_t < 0\ , \ \text{ for some } 0 \leq t \leq T).$$

Simulate the above process for $u = 50$, $\lambda = 1$, $F(z) = 1 - e^{-z}$ (i.e. claims are exponentially distributed with mean 1) and the time horizon is $T = 1000$ in order to estimate the finite horizon ruin probability $\Psi(50, 1000)$ when $\rho = 0.02,\ 0.04,\ 0.06,\ 0.08,\ 0.1$ together with 95% confidence intervals.

Problem 3. Consider the following discrete time model. Individual claims are independent random variables with distribution $F(z)$. During the kth period a random number of claims, N_k, occurs. We denote by $Z_{i,k}$ the ith claim of the kth period. The initial capital is u and in each period there is a fixed income from premiums, $b > 0$. Thus the free reserves process is

$$\begin{aligned} X_0 &= u \\ X_k &= X_{k-1} + b - \sum_{i=1}^{N_k} Z_{i,k}, \qquad k = 1, 2, \ldots, K. \end{aligned}$$

The ruin probability is $\Psi = \mathbb{P}(X_k < 0, \text{ for some } \leq k \leq K)$. Suppose that the claims have exponential with mean 1 and the number of claims in each period is negative binomial:

$$\mathbb{P}(N_k = i) = \binom{r + i - 1}{i} p^r (1 - p)^i, \qquad i = 0, 1, 2, \ldots.$$

Suppose that $p = 0.1$ and $r = 5$. Also $b = 50$ and $K = 6$.

a) Estimate the ruin probability, together with 95% confidence intervals for $u = 20, 30, 40, 50, 60$.

b) Use a multivariate normal approximation in order to estimate the same probability and compare your results. Hint: Compute analytically the mean and covariance of the random vector $(X_1, X_2, \ldots, X_6)$ and consider the random vector of *Gaussian* random variables $(Y_1, Y_2, \ldots, Y_6)$. Simulate the random vector Y and estimate the probability $\mathbb{P}(Y_k < 0, \text{ for some } k = 1, 2, \ldots, 6)$.