Εργασία : Μορφές Σύγκλησης Τυχαίων Μεταβλητών

Ασκήσεις

**Example 14.1.13.** *(Sequence waiting time.)* Let $\{r_n\}_{n\geq 1}$ be infinite fair coin tossing (cf. Subsection 2.6), and let $\tau = \inf\{n \geq 3 : r_{n-2} = 1, r_{n-1} = 0, r_n = 1\}$ be the first time the sequence heads-tails-heads is completed. Surprisingly, $\mathbf{E}(\tau)$ can be computed using martingales. Indeed, suppose that at each time $n$, a new player appears and bets \$1 on tails, then if they win they bet \$2 on heads, then if they win again they bet \$4 on heads. (They stop betting as soon as they either lose once or win three bets in a row.) Let $S_n$ be the total amount won by all the betters by time $n$. Then by construction, $\{S_n\}$ is a martingale with stopping time $\tau$, and furthermore $|S_n - S_{n-1}| \leq 7 < \infty$. Also. $\mathbf{E}(\tau) < \infty$, and $S_\tau = -\tau + 10$ by Exercise 14.4.12. Hence, $0 = \mathbf{E}(S_\tau) = -\mathbf{E}(\tau) + 10$, whence $\mathbf{E}(\tau) = 10$.

**Exercise 14.4.12.** Let $\{S_n\}$ and $\tau$ be as in Example 14.1.13.
**(a)** Prove that $\mathbf{E}(\tau) < \infty$. [Hint: Show that $\mathbf{P}(\tau \geq 3m) \leq (7/8)^m$, and

**(b)** Prove that $S_\tau = -\tau + 10$. [Hint: By considering the $\tau$ different players one at a time, argue that $S_\tau = (\tau - 3)(-1) + 7 - 1 + 1$.]

For a second example, suppose $X_0 = 50$ and that $p_{ij} = \frac{1}{2\min(i,\,100-i)+1}$ for $|j - i| \leq \min(i, 100 - i)$. This Markov chain lives on $\{0, 1, 2, \ldots, 100\}$, at each stage jumping uniformly to one of the points within $\min(i,\, 100 - i)$ of its current position. $\{X_n\} \to X$ a.s. for some random variable $X$, and indeed it is easily seen that $\mathbf{P}(X = 0) = \mathbf{P}(X = 100) = \frac{1}{2}$.

For a third example, let $S = \{2^n; n \in \mathbf{Z}\}$, with a Markov chain on $S$ having transition probabilities $p_{i,2i} = \frac{1}{3}$, $p_{i,\frac{i}{2}} = \frac{2}{3}$. This is again a martingale, and in fact it converges to 0 a.s. (even though it is unbounded).

For a fourth example, let $S = \mathbf{N}$ be the set of positive integers, with $X_0 = 1$. Let $p_{ii} = 1$ for $i$ even, with $p_{i,i-1} = 2/3$ and $p_{i,i+2} = 1/3$ for $i$ odd. Then it is easy to see that this Markov chain is a non-negative martingale, which converges a.s. to a random variable $X$ having the property that $\mathbf{P}(X = i) > 0$ for every even non-negative integer $i$.

**Definition 2.4.** *Let $C$ and $X$ be stochastic processes. The process $C \circ X$ is* *martingale transform, where*

$$(C \circ X)_n \doteq \sum_{k=1}^{n} C_k (X_k - X_{k-1}) = \sum_{k=1}^{n} C_k \Delta X_k,$$

*when $n \geq 1$ and $(C \circ X)_0 = X_0$.*

The next theorem justifies the terminology.

**Theorem 2.1.** *Let $\mathbb{F}$ be a history, the process $X$ satisfies $X \in \mathbb{F}$ and $C$ is a predictable process.*

- *If in addition $0 \leq C_n(\omega) \leq K$ and $X$ is a supermartingale, then $Y \doteq (C \circ X)$ is a supermartingale.*
- *If in addition $|C_n(\omega)| \leq K$ and $X$ is a martingale, then $Y \doteq (C \circ X)$ is a martingale.*

- Show that the $L_2$ norm of Y is equal to $L_2$ norm of X.