Project 1 – Simulation of Risk Process

Consider a risk process where claims occur according to a Poisson process with rate λ and their sizes are i.i.d. non-negative random variables, $\{Z_k\}$, $k = 1, 2, \ldots$, with distribution F. Let $\mu := \int_0^\infty x dF(x)$ denote its mean. The rate at which income from premium payments accumulates is assumed constant and equal to c and the initial capital is u. Thus the free reserves of the company at time t are given by

$$X_t = u + ct - \sum_{k=1}^{N_t} Z_k.$$

We assume that $c > \lambda\mu$ which means that the rate at which the reserves increase on the average due to premium accumulation is greater than the rate at which they decrease due to the occurrence of claims.

Define the finite horizon ruin probability as $\Psi(u, T) := \mathbb{P}(\inf_{0 \le t \le T} X_t < 0)$, i.e. the probability that the free reserves become negative at some point in the interval $[0, T]$. When the time horizon T is large the finite ruin probability can be approximated by the infinite horizon ruin probability $\Psi(u) := \mathbb{P}(\inf_{t \ge 0} X_t < 0)$.

Suppose that the claim distribution has density $f(x) = \frac{1}{2}x^2 e^{-x}$, $\lambda = 2$, $u = 20$, and $c = 6.5$. Write a program that simulates the risk process and estimate the infinite time ruin probability (taking $T = 1000$) by performing a large number of independent runs (each of duration T). Obtain 95% confidence intervals for the ruin probability.

Project 2 – Simulation of a Gaussian Process

Let $\{X_t; t \in [0,T]\}$ be a gaussian process with zero mean and covariance matrix $R(s,t)$. Simulate the process by discretizing the interval $[0,T]$ to

$$0, T/N, 2T/N, \dots (N-1)T/N, T.$$

Set $t_i := iT/N$ and consider the symmetric $(N+1)\times(N+1)$ matrix R_N whose (i,j) element is $R(t_i, t_j)$. Consider the following two cases:

1) $R(s,t) = \min(s,t)$. (Brownian Motion)

2) $R(s,t) = e^{-|t-s|}$. (Ornstein-Uhlenbeck Process)

For these two cases take $T = 10$ and discretize the interval $[0,10]$ into 101 points. Consider the decompositions $R_N = \Phi_N \Lambda_N \Phi_N^\top$ (spectral representation) and $R_N = L_N L_N^\top$ (Cholesky decomposition: here L_N is lower triangular) and use them to simulate the two processes. If $\{X_1(t); t \in [0,T]\}$ and $\{X_2(t); t \in [0,T]\}$ are a Brownian motion and an Ornstein-Uhlenbeck respectively estimate

$$\mathbb{E}\left[\int_0^T \mathbf{1}(X_1(t) > 3)dt\right] \quad \text{and} \quad \mathbb{P}(\sup_{0\le t\le T} |X_2(t)| > 2).$$

Project 3 – Poisson Shot Noise

Suppose that $\{N_t; t \geq 0\}$ is a Poisson process with rate λ and denote its points by T_k, $k = 1, 2, \ldots$. Let $\{Z_k; k \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with distribution function F and moment generating function $M(\theta) := \mathbb{E}[e^{\theta Z_1}]$. Let $\alpha > 0$ and define the function

$$h(x) := \begin{cases} e^{-\alpha x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Consider the process $\{X_t; t \geq 0\}$ with

$$X_t = \sum_{k=1}^{\infty} Z_k h(t - T_k) \quad t \geq 0.$$

(Empty sums are by definition equal to 0.)

1) What is the mean $\mathbb{E}[X_t]$?

2) What is the variance $\mathsf{Var}(X_t)$?

3) What is the covariance $\mathbb{E}[X_t X_{t+\tau}]$, $t, \tau > 0$?

4) Can you evaluate the moment generating function $\mathbb{E}[e^{\theta X_t}]$?

5) In all of the above cases evaluate the limit as $t \to \infty$.

Hint: One way of proceeding is to consider the interval $[0, t]$ and to condition on the number of points of the Poisson process in it. Conditional on the number of points, each one of them is uniformly distributed in $[0, t]$, independently of the others.

Project 4 – Compound Poisson Process

Suppose that $\lambda(t)$, $t \geq 0$ is a continuous nonnegative function and $\{N_t; t \geq 0\}$ is a time-varying Poisson process with rate $\lambda(t)$. Let Z_k, $k = 1, 2, \ldots$, be i.i.d. random variables with distribution F and characteristic function $\Phi(u) := \mathbb{E}[e^{iuZ_1}]$. Consider the compound Poisson process $\{X_t; t \geq 0\}$ where

$$X_t = \sum_{k=1}^{N_t} Z_k.$$

(As usual, empty sums are assumed equal to 0.)

1) Argue that, conditional on the number of points in the interval $[0, t]$ being equal to n, the unordered points of the time varying Poisson process are independent random variables with density $\frac{\lambda(s)}{\int_0^t \lambda(x)dx}$, $s \in [0, t]$.

2) Use this to compute $\mathbb{E}[X_t]$ and $\mathsf{Var}(X_t)$.

3) Use the independent increments property of the Poisson process to compute $\mathsf{Cov}(X_s, X_t)$.

4) If $\{Z_k\}$ are Bernoulli random variables with $\mathbb{P}(Z_1 = +1) = \mathbb{P}(Z_1 = -1) = 1/2$ determine the characteristic function of X_t.