
Problem 1. Let $\{X_k\}$, $k = 1, 2, \ldots$, be independent, identically distributed random variables with distribution $P(X_k = -1) = 1/4$, $P(X_k = 0) = 1/2$, $P(X_k = 1) = 1/4$. Set $S_n = X_1 + X_2 + \cdots + X_n$, $S_0 = 0$. Finally, let $T = \min\{n : S_n \in \{-1, 5\}\}$, i.e. T is the first time that S_n becomes either -1 or 5.

i) Argue that S_n is a martingale.

ii) Use the Optional Sampling Theorem to compute the probability $P(S_T = 5)$

iii) Show that the process $Y_n := S_n^2 - \frac{n}{2}$, $n = 1, 2, \ldots$ is a martingale.

iv) Use once more the Optional Sampling Theorem for the process Y_n to compute ET.

Problem 2. Suppose that $\{\xi_i\}$, $i = 0, 1, 2, \ldots$ are i.i.d. random variables with $P(\xi_0 = 1) = P(\xi_0 = -1) = 1/2$. Let $X_n = \sum_{k=1}^{n} \xi_{k-1}\xi_k$ and $\mathcal{F}_n = \sigma - \{\xi_0, \xi_1, \ldots, \xi_n\}$ for $n = 1, 2, 3, \ldots$.

a) Show that X_n is an $\mathcal{F}$–martingale.

b) Define $Y_n = X_n^2 - n$. Show that Y_n is also a $\mathcal{F}_n$–martingale.

c) Let $T = \inf\{n \in \mathbb{N} : |X_n| = 10\}$. Compute ET.

Problem 3. Let $\{W_t; t \geq 0\}$ be brownian motion with mean 0 and variance constant σ^2.

i) Suppose that $W_0 = x$ and let $T = \inf\{t \geq 0 : W_t = a \text{ or } W_t = b\}$, i.e. T is the hitting time of the set $\{a, b\}$. Use the fact that W_t is a martingale, together with the optional sampling theorem to obtain an expression for the probabilities $p_a = P(W_T = a)$, $p_b = P(W_T = b)$.

ii) Use the fact that $W_t^2 - \sigma^2 t$ is also a martingale to obtain an expression for ET.

iii) Show that $W_t^3 - 3\sigma^2 t W_t$ is a martingale.

iv) Let $m_a = E[T|W_T = a]$, $m_b = E[T|W_T = b]$. Use the fact that $W_t^3 - \sigma^2 t$ is a martingale and the optional sampling theorem to compute m_a, m_b. (Hint: You may also need to use the fact that $ET = p_a m_a + p_b m_b$.)

Problem 4. Consider a geometric Brownian motion defined as $X_t = e^{W_t}$ where W_t is standard Brownian motion.
Let $x > 0$ a given level and $T_x = \inf\{t > 0 : X(t) = x\}$. Thus T_x is the first time the process $X(t)$ reaches the level x.

i) Compute $\mathrm{Cov}(X_s, X_t)$ $(s < t)$.

ii) Evaluate $P(X_1 > 2)$.

iii) What is the probability that $T_{1/2} < T_2$ i.e. the probability that $\{X_t\}$ reaches level 1/2 before it reaches level 2? (Hint: Transform this problem into one about hitting times of Brownian motion.)

iv) What is the mean of the random variable T_x?

v) Compute the Laplace transform of T_x, Ee^{-sT_x}.

Problem 5. Let $X_t = \int_0^t e^{W_s} dW_s$ where $\{W_t\}$ is standard brownian motion and the stochastic integral is defined in the Itô sense. What is EX_t, $\mathrm{Var}(X_t)$ and $\mathrm{Cov}(X_s, X_t)$ where $s < t$? Compute the same quantities for the process $Y_t = \int_0^t e^s dW_s$. If $s < t$ compute $E[Y_s|Y_t]$.

Problem 6. If W_t is standard Brownian motion, use the Itô formula to compute the stochastic integrals

i) $\int_0^t W_s^n dW_s$, n positive integer.

ii) $\int_0^t e^{\beta W_s} dW_s$, $\beta \in \mathbb{R}$.

iii) $\int_0^t \frac{1}{1+W_s^2} dW_s$.

Problem 7. Let $\{W_t; t \geq 0\}$ denote the standard brownian motion and $X_t := W_t - tW_1$, for $t \in [0, 1]$ denote the standard *brownian bridge*. Compute the expected area under the graph of the standard brownian bridge, i.e. $E \int_0^1 X_t dt$, the *expected absolute area under the graph,* $E \int_0^1 |X_t| dt$, and the variance of the area $\mathrm{Var}\left(\int_0^1 X_t dt\right)$. (Hint: In many instances it pays to change the order of integration and expectation.)

Problem 8. Simulate the standard brownian bridge. (You may use any method you like but the representation $X_t = W_t - tW_1$, $t \in [0, 1]$ may be the easiest to implement.) Use a Monte Carlo estimator to estimate

$$P(\max_{0 \leq t \leq 1} |X_t| > 2) \quad \text{and} \quad E \max_{0 \leq t \leq 1} |X_t|.$$