MSc MATHS ECON - Tutorial 4

Differential Equations & Systems

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First Order Linear Differential Equations

Differential equation is an equation that relates one or more functions of time and their derivatives.
Solve $y'(t) + by(t) = 0$

$$y' + by = 0 \Rightarrow \frac{y'}{y} = -b$$

observe that $\frac{y'}{y} = \frac{dlny}{dt}$. So, integrating both sides we have:

$$lny = -bt + C$$

$$y(t) = e^{-bt+C} = e^{-bt}e^{C}$$

where C is an arbitrary constant and putting $e^C = c$ we obtain

$$y = ce^{-bt}$$

Consider the following FODE:

$$a_0 y'(t) + a_1 y(t) = g(t)$$

There are many ways to solve this non-homogeneous equation. We will first find the characteristic polynomial to determine the roots of the equation and then, will use the Lagrange's method of variation of parameters to determine the general solution including the particular integral.
Firstly, bring the original equations in normal form:

$$y'(t) + \frac{a_1}{a_0} y(t) = a_0^{-1} g(t)$$

$$y'(t) + by(t) = a_0^{-1} g(t)$$

where $b = \frac{a_1}{a_0}$.

Next, find the characteristic root of the homogeneous equation. Letting:

$$y(t) = e^{\lambda t} \qquad y'(t) = \lambda e^{\lambda t}$$

the homogeneous equation becomes:

$$\lambda e^{\lambda t} + b e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda + b) = 0$$

Since $e^{\lambda t} \neq 0$

$$\lambda = -b$$

Thus the solution of the homogeneous equation is:

$$y^h(t) = ce^{\lambda t} = ce^{-bt}$$

where c is an arbitrary constant. The solution will be stable (damped movement) as long as:

$$-b < 0$$

Apply Lagrange's method of variation of parameters. Let the following function, with time-varying coefficient $c(t)$ be a solution to the original non-homogeneous equation:

$$y(t) = c(t)e^{-bt}$$

Taking the time derivative of this expressions:

$$y'(t) = c'(t)e^{-bt} - bc(t)e^{-bt}$$

and substituting into the original $y'(t) + by(t) = a_0^{-1}g(t)$ yields:

$$c'(t)e^{-bt} - bc(t)e^{-bt} + bc(t)e^{-bt} = a_0^{-1}g(t)$$

$$c'(t)e^{-bt} = a_0^{-1}g(t)$$
$$c'(t) = e^{bt}a_0^{-1}g(t)$$

First Order Linear Differential Equations

Integrate w.r.t. time yields (without the constant of integration) a solution for $c(t)$:

$$c(t) = \int e^{bt} a_0^{-1} g(t)dt = \frac{1}{a_0} \int e^{bt} g(t)dt$$

Hence, the particular solution is:

$$\overline{y}(t) = \frac{1}{a_0} e^{-bt} \int e^{bt} g(t)dt$$

We can write the general solution as:

$$y(t) = y^h(t) + \overline{y}(t) = ce^{-bt} + \frac{1}{a_0} e^{-bt} \int e^{bt} g(t)dt$$

Finally, an initial condition is required to determine arbitrary constant c.

Example

Solve the following non-homogeneous equation:

$$3y'(t) + 6y(t) = e^{-2t}$$

subject to: $y(0) = 1$.
In normal form:

$$y'(t) + 2y(t) = \frac{1}{3}e^{-2t}$$

Find the characteristic root of the homogeneous equation:

$$\lambda + 2 = 0 \Leftrightarrow \lambda = -2$$

Thus the solution of the homogeneous equations is:

$$y^h(t) = ce^{-2t}$$

Next, apply the method of variation of parameters:

$$y(t) = c(t)e^{-2t}$$

Taking the time derivative of this expressions:

$$y'(t) = c'(t)e^{-2t} - 2c(t)e^{-2t}$$

and substituting into the original $y'(t) + 2y(t) = \frac{1}{3}e^{-2t}$ yields:

$$(c'(t)e^{-2t} - 2c(t)e^{-2t}) + 2c(t)e^{-2t} = \frac{1}{3}e^{-2t}$$

$$c'(t)e^{-2t} = \frac{1}{3}e^{-2t}$$

$$c'(t) = \frac{1}{3}$$

$$c(t) = \frac{1}{3}t$$

Hence, the particular solution is:

$$\overline{y}(t) = \frac{1}{3}te^{-2t}$$

Method of undetermined coefficients

We will solve the non-homogeneous equation $y'(t) + 2y(t) = \frac{1}{3}e^{-2t}$ by trying a solution in the form of a first order exponential guess function in t.

$$y_t = ae^{-2t}, \quad y_t' = -2ae^{-2t}$$

Substitution into the the non-homogeneous equation yields:

$$-2ae^{-2t} + 2ae^{-2t} = \frac{1}{3}e^{-2t}$$

but a remains undetermined. Try out an alternative guess function y_t (multiplying the guess function by t):

$$y_t = ate^{-2t}, \quad y_t' = ae^{-2t} - 2ate^{-2t}$$

Substitution into the the non-homogeneous equation yields:

$$ae^{-2t} - 2ate^{-2t} + 2ate^{-2t} = \frac{1}{3}e^{-2t} \Rightarrow ae^{-2t} = \frac{1}{3}e^{-2t} \Rightarrow a = \frac{1}{3}$$

So, the particular solution is:

$$\overline{y}(t) = \frac{1}{3}te^{-2t}$$

General Solution

Hence, the general solution is:

$$y(t) = y^h(t) + \overline{y}(t) = ce^{-2t} + \frac{1}{3}te^{-2t}$$

Use the initial condition $y(0) = 1$ to determine c:

$$c = 1$$

Therefore:

$$y(t) = e^{-2t} + \frac{1}{3}te^{-2t}$$

Second Order Linear Differential Equations

Consider the following linear second order linear differential equation:

$$a_0 y''(t) + a_1 y'(t) + a_2 y(t) = g(t)$$

Firstly, bring the original equation in normal form:

$$y''(t) + \frac{a_1}{a_0} y'(t) + \frac{a_2}{a_0} y(t) = \frac{1}{a_0} g(t)$$

$$y''(t) + b_1 y'(t) + b_2 y(t) = \frac{1}{a_0} g(t)$$

where $b_1 = \frac{a_1}{a_0}$ and $b_2 = \frac{a_2}{a_0}$. Next, find the characteristic root of the homogeneous equation. Letting:

$$y(t) = e^{\lambda t} \qquad y'(t) = \lambda e^{\lambda t} \qquad y''(t) = \lambda^2 e^{\lambda t}$$

the homogeneous equation becomes:

$$\lambda^2 e^{\lambda t} + b_1 \lambda e^{\lambda t} + b_2 e^{\lambda t} = 0$$

$$e^{\lambda t}(\lambda^2 + b_1 \lambda + b_2) = 0$$

Since $e^{\lambda t} \neq 0$ we have that:

$$\lambda^2 + b_1\lambda + b_2 = 0$$

The quality of the roots depends on whether:

$$\Delta \lesseqgtr 0$$

The solution will be stable (monotonic convergent movement as $t \to \infty$) as long as:

$$\lambda_i < 0, \ \forall i = 1, 2$$

In general, whatever Δ, the NASC for stability i.e. for the real part of the roots be < 0 requires (Gandolfo p.198):

$$b_1 > 0 \quad and \quad b_2 > 0$$

The solution of the homogeneous equation becomes:

$$y^h(t) = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t}, \quad \Delta > 0$$

$$y^h(t) = A_1 e^{\lambda t} + A_2 t e^{\lambda t}, \quad \Delta = 0$$

$$y^h(t) = e^{at}(A_1 cosbt + A_2 sinbt), \quad \Delta < 0 \quad \lambda_{1,2} = \alpha \pm ib$$

Variation of Parameters

Next, apply the Lagrange's method of variation of parameters. Let the following function with time varying coefficients $A_1(t)$ and $A_2(t)$ be a solution of the non-homogeneous equation (assuming the case $\Delta > 0$):

$$y(t) = A_1(t)e^{\lambda_1 t} + A_2(t)e^{\lambda_2 t}$$

The above function will be a solution to the non-homogeneous equation if $A_1(t)$ and $A_2(t)$ satisfy the two equations (Gandolfo p.201):

$$\textit{Condition } 1 : A_1'(t)e^{\lambda_1 t} + A_2'(t)e^{\lambda_2 t} = 0$$

$$\textit{Condition } 2 : A_1'(t)(e^{\lambda_1 t})' + A_2'(t)(e^{\lambda_2 t})' = \frac{1}{a_0}g(t)$$

- Solve the linear system for $A_1'(t)$ and $A_2'(t)$.
- Integrating the solutions w.r.t. time yields a solution for $A_1(t)$ and $A_2(t)$. Hence you find the particular solution $\overline{y}_t$.
- We can write the general solution as $y(t) = y^h(t) + \overline{y}(t)$

Discriminant $\Delta > 0$

Solve:

$$y''(t) - 5y'(t) + 6y(t) = t$$

The equation is already in normal form:

$$y''(t) + b_1 y'(t) + b_2 y(t) = \frac{1}{a_0} g(t)$$

with:

$$a_0 = 1,\ b_1 = -5,\ b_2 = 6,\ g(t) = t$$

The characteristic polynomial:

$$\lambda^2 - 5\lambda + 6 = 0$$

$$\Delta = 1 > 0,\ \lambda_1 = 2\ \lambda_2 = 3$$

Thus the solution of the homogeneous equations is:

$$y^h(t) = A_1 e^{2t} + A_2 e^{3t}$$

Discriminant $\Delta > 0$

Next, we will solve the non-homogeneous equation

$$y''(t) - 5y'(t) + 6y(t) = t$$

by trying a solution in the form of a first order polynomial guess function in t:

$$y_t = at + \beta, \quad y_t' = a \quad y_t'' = 0$$

and will substitute for y_t and y_t' into the original equation leaving the RHS unchanged:

$$-5a + 6(at + \beta) = t$$

$$6at + (6\beta - 5a) = t$$

Discriminant $\Delta > 0$

Equating coefficients on both sides yields a linear system in two unknown parameters (the undetermined coefficients):

$$\left\{ \begin{array}{c} 6a = 1 \\ 6\beta - 5a = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} a = \frac{1}{6} \\ \beta = \frac{5}{36} \end{array} \right\}$$

Therefore, the solution of the non-homogeneous equation is:

$$\bar{y}_t = \frac{1}{6}t + \frac{5}{36}$$

The general solution equals the sum of the solution of the homogeneous and the non-homogeneous equation.

$$y_t = A_1 e^{2t} + A_2 e^{3t} + \frac{1}{6}t + \frac{5}{36}$$

Discriminant Δ=0

Solve:

$$y''(t) - 6y'(t) + 9y(t) = \frac{e^{3t}}{t^2}$$

subject to $y(1) = 0$, $y'(1) = 1.5e^3$. The equation is already in normal form:

$$y''(t) + b_1 y'(t) + b_2 y(t) = \frac{1}{a_0} g(t)$$

with:

$$a_0 = 1, \;\; b_1 = -6, \;\; b_2 = 9, \;\; g(t) = \frac{e^{3t}}{t^2}$$

The characteristic polynomial:

$$\lambda^2 - 6\lambda + 9 = 0 \Leftrightarrow (\lambda - 3)^2 = 0$$

Implies an unstable solution since $\lambda = \lambda_1 = \lambda_2 = 3 \not< 0$. Moreover the NASC for stability are not satisfied:

$$b_1 = -6 \not> 0, \;\; b_2 = 9 > 0$$

Thus the solution of the homogeneous equations is:

$$y^h(t) = A_1 e^{3t} + A_2 t e^{3t}$$

Discriminant Δ=0

Next, apply the Lagrange's method of variation of parameters. Let the following function with time varying coefficients $A_1(t)$ and $A_2(t)$ be a solution of the non-homogeneous equation

$$y(t) = A_1(t)e^{3t} + A_2(t)te^{3t}$$

$$Condition\ 1 : A_1'(t)e^{3t} + A_2'(t)te^{3t} = 0$$

$$Condition\ 2 : A_1'(t)(e^{3t})' + A_2'(t)(te^{3t})' = \frac{e^{3t}}{t^2} \Rightarrow$$

$$A_1'(t)(3e^{3t}) + A_2'(t)(e^{3t} + 3te^{3t}) = \frac{e^{3t}}{t^2} \Rightarrow$$

$$A_1'(t)(3e^{3t}) + A_2'(t)e^{3t}(1 + 3t) = \frac{e^{3t}}{t^2}$$

Discriminant $\Delta=0$

Solve the linear system for $A_1'(t)$ and $A_2'(t)$:

$$A_1'(t)e^{3t} + A_2'(t)te^{3t} = 0$$

$$A_1'(t)(3e^{3t}) + A_2'(t)e^{3t}(1+3t) = \frac{e^{3t}}{t^2}$$

Use Cramer's Rule to solve the 2×2 system of equations:

$$A_1'(t) = \frac{\begin{vmatrix} 0 & te^{3t} \\ \frac{e^{3t}}{t^2} & e^{3t}(1+3t) \end{vmatrix}}{\begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t}(1+3t) \end{vmatrix}} = \frac{-\frac{e^{3t}}{t^2}te^{3t}}{e^{3t}e^{3t}(1+3t) - 3e^{3t}te^{3t}} = \frac{-1}{t}$$

$$A_2'(t) = \frac{\begin{vmatrix} e^{3t} & 0 \\ 3e^{3t} & \frac{e^{3t}}{t^2} \end{vmatrix}}{\begin{vmatrix} e^{3t} & te^{3t} \\ 3e^{3t} & e^{3t}(1+3t) \end{vmatrix}} = \frac{e^{3t}\frac{e^{3t}}{t^2}}{e^{3t}e^{3t}(1+3t) - 3e^{3t}te^{3t}} = \frac{1}{t^2}$$

Discriminant Δ=0

Integrating w.r.t time yields a solution for $A_1(t)$ and $A_2(t)$:

$$A_1(t) = \int \frac{-1}{t} dt = -lnt$$

$$A_2(t) = \int \frac{1}{t^2} dt = -\frac{1}{t}$$

Hence the particular solution is:

$$\overline{y}(t) = -lnte^{3t} - \frac{1}{t}te^{3t} = -(e^{3t}lnt + e^{3t})$$

The general solution is $y(t) = y^h(t) + \overline{y}(t)$, so

$$y(t) = A_1 e^{3t} + A_2 te^{3t} - (e^{3t}lnt + e^{3t})$$

The initial conditions $y(1) = 0$, $y'(1) = 1.5e^3$ serve to determine arbitrary constants A_1 and A_2. Hence the general solution is

$$y(t) = -e^{3t}(lnt + 1) - 1.5e^{3t} + 2.5te^{3t}$$

Discriminant $\Delta < 0$

Let us solve the following second-order differential equation:

$$y''(t) - 2y'(t) + 5y(t) = sin(t)$$

The characteristic polynomial:

$$\lambda^2 - 2\lambda + 5 = 0$$

Since $\Delta = 4 - 4 \cdot 5 = -16 < 0$, the roots of the characteristic equation are a complex conjugates:

$$\lambda_{1,2} = \frac{2 \pm \sqrt{-16}}{2} = 1 \pm 2i$$

Cartesian coordinates

$$(a, b) = (1, 2)$$

So, the solution of the homogeneous equations is:

$$y^h(t) = e^{at}(A_1 cosbt + A_2 sinbt) = e^t(A_1 cos2t + A_2 sin2t)$$

Next we will solve the non-homogeneous equation $y''(t) - 2y'(t) + 5y(t) = sin(t)$ by trying a solution in the form of a first order sinusoidal guess function in t:

$$y_t = a\, cos(\omega t) + \beta\, sin(\omega t)$$

$$y_t' = -a\omega\, sin(\omega t) + \beta\omega\, cos(\omega t)$$
$$y_t'' = -a\omega^2\, cos(\omega t) - \beta\omega^2\, sin(\omega t)$$

Therefore $-a\omega^2\, cos(\omega t) - \beta\omega^2\, sin(\omega t) - 2(-a\omega\, sin(\omega t) + \beta\omega\, cos(\omega t)) + 5(a\, cos(\omega t) + \beta\, sin(\omega t)) = sin(t)$

$$(-a\omega^2 - 2\beta\omega + 5a)cos(\omega t) + (-\beta\omega^2 + 2a\omega + 5\beta)sin(\omega t) = sin(t)$$

Discriminant $\Delta < 0$

Equating coefficients and angles on both sides yields:

$$\left\{ \begin{array}{c} \omega = 1 \\ -a\omega^2 - 2\beta\omega + 5a = 0 \\ -\beta\omega^2 + 2a\omega + 5\beta = 1 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \omega = 1 \\ -2\beta + 4a = 0 \\ 2a + 4\beta = 1 \end{array} \right\} \Leftrightarrow$$

$$\left\{ \begin{array}{c} \omega = 1 \\ a = \frac{1}{10} \\ \beta = \frac{1}{5} \end{array} \right\}$$

Therefore, the solution of the non-homogeneous equation is:

$$\bar{y}_t = \frac{1}{10} cos(t) + \frac{1}{5} sin(t)$$

So, the general solution equals:

$$y_t = y_t^h + \bar{y}_t \Rightarrow$$

$$y_t = e^t(A_1 cos2t + A_2 sin2t) + \frac{1}{10} cos(t) + \frac{1}{5} sin(t)$$

Write the following 2nd order differential equation as a system of first order, linear differential equations.

$$y'' + ay' + by = g(t)$$

Since the order of the differential equation is $p = 2$, define $p - 1 = 1$ new variables, say x:

$$x \equiv y'$$

$$x' \equiv y''$$

Substituting in the original and solving for x' yields:

$$x' + ax + by = g(t) \Rightarrow$$

$$x' = -by - ax + g(t)$$

$$\begin{bmatrix} y' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}$$

Write the following third order differential equation as a system of first order, linear differential equations.

$$y''' + ay'' + by' + cy = g(t)$$

Since the order of the differential equation is $p = 3$, define $p - 1 = 2$ new variables, say z and x :

$$z \equiv y'$$

$$x \equiv y'' \Rightarrow x = z'$$

$$x' = y'''$$

Substituting in the original and solving for x' yields:

$$x' + ax + bz + cy = g(t) \Rightarrow$$

$$x' = -cy - bz - ax + g(t)$$

Systems of Differential Equations

$$\begin{bmatrix} y' \\ z' \\ x' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{bmatrix} \begin{bmatrix} y \\ z \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(t) \end{bmatrix}$$

Stacking yields a linear first-order $p \times p$ difference system:

$$\underset{p\times 1}{X'} = \underset{p\times p}{A}\underset{p\times 1}{X} + \underset{p\times 1}{G}$$

General case: First order $n \times n$ systems (Gandolfo p.245)

Systems of Differential Equations

Solve the following non-homogeneous system of differential equations:

$$\left\{ \begin{array}{c} x' = x + 2y + 3t \\ y' = 2x + y + 2 \end{array} \right\}$$

This is a linear 2×2 system in x and y, already in normal. In matrix form:

$$X' = AX + g$$

where

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad X' := \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad X := \begin{bmatrix} x \\ y \end{bmatrix}$$

$$g(t) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Real distinct roots ($\Delta>0$)

Apply the direct method and find the characteristic polynomial of square matrix A:

$$\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = 0 \Rightarrow (1-\lambda)(1-\lambda)-4=0 \Rightarrow \lambda^2-2\lambda-3=0$$

Since $\Delta = 16 > 0$ the polynomial has a pair of real distinct roots (eigenvalues), $\lambda_1 = 3$, $\lambda_2 = -1$. Since the eigenvalues are distinct, the eigenvector $v_1 = (v_{11}, v_{12})$, $v_2 = (v_{21}, v_{22})$ will be linear independent:

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -2v_{11}+2v_{12}=0 \\ 2v_{11}-2v_{12}=0 \end{array}$$

So, we have that $v_{11} = v_{12}$.Therefore

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{12} \\ v_{12} \end{bmatrix} = v_{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, an eigenvector corresponding to $\lambda_1 = 3$ is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

while the eigenvector corresponding to $\lambda_2 = -1$ is

$$\begin{bmatrix} 1-(-1) & 2 \\ 2 & 1-(-1) \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 2v_{21} + 2v_{22} = 0 \\ 2v_{21} + 2v_{22} = 0 \end{matrix}$$

So, we have that $v_{21} = -v_{22}$. Therefore

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} -v_{22} \\ v_{22} \end{bmatrix} = v_{22} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So, an eigenvector corresponding to $\lambda_2 = -1$ is

$$v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

The solution of the homogeneous system is:

$$X = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$$

Real distinct roots ($\Delta > 0$)

$$X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

$$\left\{ \begin{array}{l} x_t = c_1 e^{3t} - c_2 e^{-t} \\ y_t = c_1 e^{3t} + c_2 e^{-t} \end{array} \right\}$$

where c_1 and c_2 are arbitrary constants.
We can now turn to the problem of finding a particular solution of the non-homogeneous system. The method of undetermined coefficients can be applied here too. Since

$$g(t) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

let us try

$$\overline{X}(t) = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} t + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

where α_1, α_2, β_1, β_2 are undetermined constants.

Real distinct roots (Δ>0)

$$\overline{X}' = A\overline{X} + g$$

$$g = \overline{X}' - A\overline{X}$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} t + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right)$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} - \begin{bmatrix} \alpha_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} t - \begin{bmatrix} \beta_1 + 2\beta_2 \\ 2\beta_1 + \beta_2 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = - \begin{bmatrix} \alpha_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} t + \begin{bmatrix} \alpha_1 - \beta_1 - 2\beta_2 \\ \alpha_2 - 2\beta_1 - \beta_2 \end{bmatrix}$$

Real distinct roots ($\Delta > 0$)

Clearly requiring that a_1 , a_2 , b_1 and b_2 satisfy the two linear algebraic systems

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} \alpha_1 + 2a_2 \\ 2a_1 + a_2 \end{bmatrix} \qquad \textit{and} \qquad \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \alpha_1 - \beta_1 - 2\beta_2 \\ \alpha_2 - 2\beta_1 - \beta_2 \end{bmatrix}$$

whose solution is easily found to be

$$a_1 = 1, \;\; a_2 = -2, \;\; b_1 = -3, \;\; b_2 = 2$$

So, the particular solution is:

$$\overline{X}(t) = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Therefore, $GS = CF + PS$

$$X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} t + \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\left\{ \begin{array}{c} x_t = c_1 e^{3t} - c_2 e^{-t} + t - 3 \\ y_t = c_1 e^{3t} + c_2 e^{-t} - 2t + 2 \end{array} \right\}$$

Real repeated roots (Δ=0)

Solve the following differential system:

$$\left\{ \begin{array}{c} x' = -4x + y \\ y' = -1x - 2y \end{array} \right\}$$

This is a linear 2×2 system in x and y, already in normal. In matrix form:

$$X' = AX + g$$

where

$$A := \begin{bmatrix} -4 & 1 \\ -1 & -2 \end{bmatrix}, \quad X' := \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad X := \begin{bmatrix} x \\ y \end{bmatrix}, \quad g(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Apply the direct method and find the characteristic polynomial of square matrix A:

$$\begin{bmatrix} -4-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} = 0 \Rightarrow (-4-\lambda)(-2-\lambda)+1 = 0 \Rightarrow \lambda^2+6\lambda+9 = 0$$

Since $\Delta = 0$ the polynomial has a real repeated root (eigenvalue), $\lambda = \lambda_1 = \lambda_2 = -3$.

Real repeated roots (Δ=0)

The independent eigenvector v_1 is:

$$\begin{bmatrix} -4-(-3) & 1 \\ -1 & -2-(-3) \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{c} -v_{11}+v_{12}=0 \\ -v_{11}+v_{12}=0 \end{array}$$

So, we have that $v_{11} = v_{12}$. Therefore

$$v_1 = \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} v_{12} \\ v_{12} \end{bmatrix} = v_{12} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, the independent eigenvector corresponding to $\lambda_1 = -3$ is

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Real repeated roots (Δ=0)

The generalized eigenvector v_2 is

$$(A - \lambda \mathrm{I})v_2 = v_1$$

$$\begin{bmatrix} -4-(-3) & 1 \\ -1 & -2-(-3) \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{matrix} -v_{21} + v_{22} = 1 \\ -v_{21} + v_{22} = 1 \end{matrix} \rightarrow v_{21} = v_{22} - 1$$

Therefore, the generalized eigenvector is:

$$v_2 = \begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} v_{22} - 1 \\ v_{22} \end{bmatrix} \overset{v_{22}=1}{=} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Real repeated roots (Δ=0)

The solution of the homogeneous system when $\Delta = 0$ is:

$$X = c_1 v_1 e^{\lambda t} + c_2(t v_1 + v_2) e^{\lambda t}$$

The solution of the system is:

$$X = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 (t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}) e^{-3t}$$

$$\left\{ \begin{array}{c} x_t = c_1 e^{-3t} + c_2 t e^{-3t} \\ y_t = c_1 e^{-3t} + c_2 (t+1) e^{-3t} \end{array} \right\}$$

Complex eigenvalues (Δ<0) (for those interested!)

Complex eigenvalues: $\lambda_1, \lambda_2 = u + vi$

$$\left\{ \begin{array}{c} x_1(t) = e^{ut}[c_1(\gamma_1 cos(vt) - \gamma_2 sin(vt)) + c_2(\gamma_1 sin(vt) + \gamma_2 cos(vt))] \\ x_2(t) = e^{ut}[c_1(\delta_1 cos(vt) - \delta_2 sin(vt)) + c_2(\delta_1 sin(vt) + \delta_2 cos(vt))] \end{array} \right\}$$

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 + i\gamma_2 \\ \delta_1 + i\delta_2 \end{pmatrix}$$

For more details for the case of complex eigenvalues for matrix A see any good book in differential equations e.g. M. Hirsch ans S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, 1974, or Morris W. Hirsch Stephen Smale and Robert L. Devaney, Differential Equations, Dynamical Systems, and an Introduction to Chaos, Elsevier, 2004