MSc MATHS ECON - TUTORIAL 2

Difference Equations and Lag Operators

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First-order Difference Equations

The general form is:

$$b_1 y_t + b_0 y_{t-1} = g(t)$$

where $g(t)$ is a known function. The solution of the homogeneous equations $b_1 y_t + b_0 y_{t-1} = 0$ is:

$$y_t^h = c(-\lambda)^t$$

where c is an arbitrary constant and $\lambda = \frac{b_0}{b_1}$.
Use the method of undetermined coefficients to find a particular solution $\bar{y}_t$. The general solution equals the sum of the solutions of the homogeneous and the non-homogeneous equation (the particular solution):

$$y_t = y_t^h + \bar{y}_t = c(-\lambda)^t + \bar{y}_t$$

In order to determine the arbitrary constant we need one additional condition:

$$y_t = y^* \quad for \quad t = t^*$$

where y^* and t^* are known values.

Polynomial guess

Solve $y_{t+1} - 5y_t = 3t + 2$ subject to $y_0 = 0$
We will start by solving the homogeneous equation $y_{t+1} - 5y_t = 0$
Let non-trivial solution $y_t = \lambda^t \neq 0$ and substitute into the homogeneous equation:

$$y_{t+1} - 5y_t = 0 \Rightarrow \lambda^{t+1} - 5\lambda^t = 0 \Leftrightarrow$$

$$\lambda^t(\lambda - 5) = 0 \Leftrightarrow \lambda = 5$$

Therefore, the solution of the homogeneous equation is:

$$y_t^h = c\lambda^t = c5^t$$

Next, we will solve the non-homogeneous equation by trying a solution in the form of a first order polynomial guess function in t:

$$y_t = at + \beta, \quad y_{t+1} = a(t+1) + \beta$$

and will substitute for y_t and y_{t+1} into the original equation leaving the RHS unchanged:

Polynomial guess

$$y_{t+1} - 5y_t = 3t + 2 \Rightarrow a(t+1) + \beta - 5(at + \beta) = 3t + 2 \Leftrightarrow$$

$$at + a + \beta - 5at - 5\beta = 3t + 2 \Leftrightarrow$$

$$-4at - 4\beta + \alpha = 3t + 2$$

Equating coefficients on both sides yields a linear system in two unknown parameters (the undetermined coefficients):

$$\left\{ \begin{array}{c} -4a = 3 \\ -4\beta + a = 2 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} a = -\frac{3}{4} \\ \beta = -\frac{11}{16} \end{array} \right\}$$

Therefore, the solution of the non-homogeneous equation is:

$$\bar{y}_t = -\frac{3}{4}t - \frac{11}{16}$$

The general solution equals the sum of the solution of the homogeneous and the non-homogeneous equation.

Polynomial guess

$$y_t = y_t^h + \bar{y}_t \Rightarrow$$

$$y_t = c5^t - \frac{3}{4}t - \frac{11}{16} \quad \forall t$$

We may use the initial condition in order to determine the arbitrary constant c in the general solution:

$$0 = y_0 = c5^0 - \frac{3}{4}0 - \frac{11}{16} \Leftrightarrow$$

$$c = \frac{11}{16}$$

Therefore, the general solution equals:

$$y_t = \frac{11}{16}5^t - \frac{3}{4}t - \frac{11}{16} \quad \forall t$$

Since the characteristic root is $\lambda = 5 > 0$ and $|\lambda| > 1$; the movement will be monotonically divergent (from the long run equilibrium) as $t \to \infty$.

Exponential guess

Solve $y_t - 3y_{t-1} = 4^t$ subject to $y_0 = 0$.
We will start by solving the homogeneous equation $y_t - 3y_{t-1} = 0$.
Let non-trivial solution $y_t = \lambda^t \neq 0$ and substitute into the homogeneous equation:

$$y_t - 3y_{t-1} = 0 \Rightarrow \lambda^t - 3\lambda^{t-1} = 0 \Leftrightarrow$$

$$\lambda^{t-1}(\lambda - 3) = 0 \Leftrightarrow \lambda = 3$$

Therefore, the solution of the homogeneous equation is:

$$y_t^h = c\lambda^t = c3^t$$

Since the characteristic root is $\lambda = 3 > 0$ and $|\lambda| > 1$; the movement will be monotonically divergent as $t \to \infty$.
Next, we will solve the non-homogeneous equation by trying a solution in the form of a first order exponential guess function in t:

$$y_t = a4^t, \quad y_{t-1} = a4^{t-1}$$

Exponential guess

Substitute for y_t and y_{t-1} into the original equation leaving the RHS unchanged:

$$y_t - 3y_{t-1} = 4^t \Rightarrow a4^t - 3a4^{t-1} = 4^t \Leftrightarrow$$

$$(a - 3a4^{-1})4^t = 4^t \Leftrightarrow a(1 - \frac{3}{4})4^t = 4^t \Leftrightarrow$$

$$a\frac{1}{4}4^t = 4^t$$

Equating coefficients on both sides allows as to determine a:

$$\frac{a}{4} = 1 \Leftrightarrow a = 4$$

Therefore, the solution of the non-homogeneous equation is:

$$\bar{y}_t = 44^t = 4^{t+1}$$

Exponential guess

The general solution equals the sum of the solution of the homogeneous and the non-homogeneous equation.

$$y_t = y_t^h + \bar{y}_t \Rightarrow$$

$$y_t = c3^t + 4^{t+1} \quad \forall t$$

We may use the initial condition in order to determine the arbitrary constant c in the general solution:

$$0 = y_0 = c3^0 + 4 \Leftrightarrow$$

$$c = -4$$

Therefore, the general solution equals:

$$y_t = -43^t + 4^{t+1} \Leftrightarrow$$

$$y_t = 4(4^t - 3^t) \quad \forall t$$

Trigonometric Formulas

$$cos(\omega t \pm \omega) = cos\omega t\ cos\omega \mp sin\omega t\ sin\omega$$

$$sin(\omega t \pm \omega) = sin\omega t\ cos\omega \pm sin\omega\ cos\omega t$$

Trigonometric guess

Solve $y_{t+1} + \frac{\sqrt{2}}{2} y_t = cos(\frac{\pi t}{4})$ subject to $y_0 = 0$.
We will start by solving the homogeneous equation
$y_{t+1} + \frac{\sqrt{2}}{2} y_t = 0$.
Let non-trivial solution $y_t = \lambda^t \neq 0$ and substitute into the homogeneous equation:

$$y_{t+1}+\frac{\sqrt{2}}{2}y_t = 0 \Rightarrow \lambda^{t+1}+\frac{\sqrt{2}}{2}\lambda^t = 0 \Leftrightarrow \lambda^t(\lambda+\frac{\sqrt{2}}{2}) = 0 \Leftrightarrow \lambda = -\frac{\sqrt{2}}{2}$$

Therefore, the solution of the homogeneous equation is:

$$y_t^h = c\lambda^t = c(-\frac{\sqrt{2}}{2})^t$$

Next, we will solve the non-homogeneous equation by trying a solution in the form of a first order sinusoidal guess function in t:

$$y_t = a\,cos(\omega t) + \beta\,sin(\omega t)$$

$$y_{t+1} = a\,cos(\omega(t+1)) + \beta\,sin(\omega(t+1))$$

Trigonometric guess

$$y_{t+1} = a\,(cos(\omega t)cos\omega - sin(\omega t)sin\omega) + \beta\,(sin(\omega t)cos\omega + sin\omega\; cos(\omega t))$$

$$= a\,cos(\omega t)cos\omega - a\,sin(\omega t)sin\omega + \beta\,sin(\omega t)cos\omega + \beta\,sin\omega\; cos(\omega t)$$

and will substitute for y_t and y_{t+1} into the original equation leaving the RHS unchanged:

$$y_{t+1} + \frac{\sqrt{2}}{2}y_t = cos(\frac{\pi t}{4})$$

$a\,cos(\omega t)cos\omega - a\,sin(\omega t)sin\omega + \beta\,sin(\omega t)cos\omega + \beta\,sin\omega\; cos(\omega t) + \frac{\sqrt{2}}{2}[a\,cos(\omega t) + \beta\,sin(\omega t)] = cos(\frac{\pi t}{4}) \Leftrightarrow$

$$(a\,cos\omega + \beta\,sin\omega + a\frac{\sqrt{2}}{2})cos(\omega t) + (-a\,sin\omega + \beta\,cos\omega + \beta\frac{\sqrt{2}}{2})sin(\omega t) = cos(\frac{\pi t}{4})$$

Trigonometric guess

Equating coefficients and angles on both sides yields:

$$\left\{\begin{array}{c} \omega = \frac{\pi}{4} \\ a\, cos\omega + \beta\, sin\omega + a\frac{\sqrt{2}}{2} = 1 \\ \beta\, cos\omega - a\, sin\omega + \beta\frac{\sqrt{2}}{2} = 0 \end{array}\right\} \Leftrightarrow$$

$$\left\{\begin{array}{c} \omega = \frac{\pi}{4} \\ a\, cos\frac{\pi}{4} + \beta\, sin\frac{\pi}{4} + a\frac{\sqrt{2}}{2} = 1 \\ \beta\, cos\frac{\pi}{4} - a\, sin\frac{\pi}{4} + \beta\frac{\sqrt{2}}{2} = 0 \end{array}\right\} \Leftrightarrow$$

$$\left\{\begin{array}{c} \omega = \frac{\pi}{4} \\ \frac{a}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} + a\frac{\sqrt{2}}{2} = 1 \\ \frac{\beta}{\sqrt{2}} - \frac{\alpha}{\sqrt{2}} + \beta\frac{\sqrt{2}}{2} = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \omega = \frac{\pi}{4} \\ a = \frac{2\sqrt{2}}{5} \\ \beta = \frac{\sqrt{2}}{5} \end{array}\right\}$$

Therefore, the solution of the non-homogeneous equation is:

$$\bar{y}_t = \frac{\sqrt{2}}{5}(2\,cos(\frac{\pi}{4}t) + sin(\frac{\pi}{4}t))$$

So, the general solution equals:

$$y_t = y_t^h + \bar{y}_t \Rightarrow$$

$$y_t = c(-\frac{\sqrt{2}}{2})^t + \frac{\sqrt{2}}{5}(2\,cos(\frac{\pi}{4}t) + sin(\frac{\pi}{4}t)) \quad \forall t$$

We may use the initial condition in order to determine the arbitrary constant c in the general solution:

$$0 = y_0 = c(-\frac{\sqrt{2}}{2})^0 + \frac{\sqrt{2}}{5}(2\,cos(\frac{\pi}{4}0) + sin(\frac{\pi}{4}0)) \Leftrightarrow$$

$$0 = c + \frac{2\sqrt{2}}{5} \Leftrightarrow c = -\frac{2\sqrt{2}}{5}$$

Trigonometric guess

Therefore, the general solution equal:

$$y_t = -\frac{2\sqrt{2}}{5}(-\frac{\sqrt{2}}{2})^t + \frac{\sqrt{2}}{5}(2\,cos(\frac{\pi}{4}t) + sin(\frac{\pi}{4}t)) \quad \forall t$$

Solve $y_{t+2} - 4y_{t+1} + 3y_t = 5^t$ subject to $y_0 = 1$ and $y_1 = 2$.
Firstly consider the homogeneous equation:
$y_{t+2} - 4y_{t+1} + 3y_t = 0$.
Its characteristic equation is:

$$\lambda^2 - 4\lambda + 3 = 0$$

$$\lambda^2 + \alpha_1\lambda + \alpha_2 = 0$$

$$\alpha_1 = -4 \qquad \alpha_2 = 3$$

The necessary and sufficient conditions for stability, namely:

$$1 + a_1 + a_2 > 0$$

$$1 - a_2 > 0$$

$$1 - a_1 + a_2 > 0$$

are not simultaneously satisfied.

Moreover, since $\Delta = 4 > 0$ and the coefficient of the characteristic polynomial alternate in sign, the roots will be positive by Descartes' Theorem. Indeed:

$$\lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 3}}{2 \cdot 1}$$

$$\lambda_1 = 3,\ \lambda_2 = 1 \qquad \lambda_1 \neq \lambda_2 \qquad \lambda_1, \lambda_2 \epsilon Re$$

Therefore, the solution of the homogeneous equation is:

$$y_t^h = A_1 \lambda_1^t + A_2 \lambda_2^t = A_1 3^t + A_2$$

Next, consider the non-homogeneous equation $y_{t+2} - 4y_{t+1} + 3y_t = 5^t$. Try an exponential guess function.

$$y_t = a5^t,\ y_{t+1} = a5^{t+1},\ y_{t+2} = a5^{t+2}$$

Substituting in the non-homogeneous equation yields:

$$a5^{t+2} - 4a5^{t+1} + 3a5^t = 5^t \Leftrightarrow 25a - 20a + 3a = 1 \Leftrightarrow a = \frac{1}{8}$$

Therefore:

$$PS: \quad \bar{y}_t = \frac{1}{8}5^t$$

$$GS: \quad y_t = A_1 3^t + A_2 + \frac{1}{8}5^t$$

Moreover:

$$IC1: \quad y_0 = 1 \Leftrightarrow A_1 3^0 + A_2 + \frac{1}{8}5^0 = 1 \Leftrightarrow A_1 + A_2 = \frac{7}{8}$$

$$IC2: \quad y_1 = 2 \Leftrightarrow A_1 3^1 + A_2 + \frac{1}{8}5^1 = 2 \Leftrightarrow 3A_1 + A_2 = \frac{11}{8}$$

Solving this linear system for $A_1$and A_2 yields: $A_1 = \frac{1}{4}, \ \ A_2 = \frac{5}{8}$
Therefore:

$$GS: \quad y_t = \frac{1}{4}3^t + \frac{5}{8} + \frac{1}{8}5^t$$

and $\underset{t \to +\infty}{lim} y_t = +\infty$

Discriminant Δ=0

Solve $y_{t+2} - 2y_{t+1} + y_t = -6^t + t$ assuming arbitrary initial conditions.

Firstly consider the HE: $y_{t+2} - 2y_{t+1} + y_t = 0$. Its characteristic equation is:

$$\lambda^2 - 2\lambda + 1 = 0 \Leftrightarrow (\lambda - 1)^2 = 0$$

Moreover $\Delta = 0$,and

$$\lambda_1 = \lambda_2 = 1 \epsilon Re \;\; with \;\; m = 2$$

where m indicates the multiplicity of the real repeated root. Hence, the solution of the homogeneous equation equals:

$$y_t^h = A_1 \lambda_1^t + A_2 t \lambda_2^t = A_1 + A_2 t$$

while the necessary and sufficient conditions for stability, namely:

$$1 + a_1 + a_2 > 0$$

$$1 - a_2 > 0$$

$$1 - a_1 + a_2 > 0$$

are not simultaneously satisfied.

Next consider the non-homogeneous equation:

$$y_{t+2} - 2y_{t+1} + y_t = -6^t + t$$

Let us try a mixed guess function for the non-homogeneous part:

$$y_t^g = (a6^t) + (\beta t + \gamma)$$

$$y_{t+1}^g = (a6^{t+1}) + (\beta(t+1) + \gamma)$$

$$y_{t+2}^g = (a6^{t+2}) + (\beta(t+2) + \gamma)$$

Substituting in the non-homogeneous equation yields:

$$a6^{t+2} + (\beta(t+2) + \gamma - 2[a6^{t+1} + \beta(t+1) + \gamma] + (a6^t) + (\beta t + \gamma) = -6^t + t$$

$$a6^{t+2} + \beta t + 2\beta + \gamma - 2a6^{t+1} - 2\beta t - 2\beta - 2\gamma + a6^t + \beta t + \gamma = -6^t + t$$

$$25a6^t = -6^t + t$$

Equating coefficients implies that $25a = -1 \Leftrightarrow a = -\frac{1}{25}$, but β and γ remain undetermined.

Discriminant Δ=0

Try out an alternative guess function y_t^{g2} (multiplying the undetermined part of of $y_t^g$by t):

$$y_t^{g2} = (a6^t) + (\beta t^2 + \gamma t)$$

$$y_{t+1}^{g2} = (a6^{t+1}) + (\beta(t+1)^2 + \gamma(t+1))$$

$$y_{t+2}^{g2} = (a6^{t+2}) + (\beta(t+2)^2 + \gamma(t+2))$$

Substituting in the non-homogeneous equation yields:
$a6^{t+2} + \beta(t+2)^2 + \gamma(t+2) - 2[a6^{t+1} + \beta(t+1)^2 + \gamma(t+1)] + a6^t + \beta t^2 + \gamma t = -6^t + t$

$$25a6^t + 2\beta = -6^t + t$$

Again, β and γ remain undetermined.

Discriminant Δ=0

Try out an alternative guess function y_t^{g3} (multiplying the undetermined part of y_t^{g2} by t):

$$y_t^{g3} = (a6^t) + (\beta t^3 + \gamma t^2)$$

$$y_{t+1}^{g3} = (a6^{t+1}) + (\beta(t+1)^3 + \gamma(t+1)^2)$$

$$y_{t+2}^{g3} = (a6^{t+2}) + (\beta(t+2)^3 + \gamma(t+2)^2)$$

Substituting in the non-homogeneous equation yields:
$(a6^{t+2}) + (\beta(t+2)^3 + \gamma(t+2)^2) - 2[(a6^{t+1}) + (\beta(t+1)^3 + \gamma(t+1)^2)] + (a6^t) + (\beta t^3 + \gamma t^2) = -6^t + t \Leftrightarrow$

$$25a6^t + 6\beta t + (6\beta + 2\gamma) = -6^t + t$$

Discriminant Δ=0

Equating coefficient yields a unique solution for β and γ as well:

$$6\beta = 1 \wedge 6\beta + 2\gamma = 0$$

$$\beta = \frac{1}{6} \wedge \gamma = -\frac{1}{2}$$

Therefore:

$$PS: \quad \bar{y}_t = -\frac{1}{25}6^t + \frac{1}{6}t^3 - \frac{1}{2}t^2$$

$$GS: \quad y_t = A_1 + A_2 t - \frac{1}{25}6^t + \frac{1}{6}t^3 - \frac{1}{2}t^2$$

Complex numbers

The imaginary number i is defined solely by the property that its square is -1.

$$i^2 = -1$$

A complex number z is a number that can be expressed in the form:

$$z = a + bi$$

where a and b are real numbers, and i represents the imaginary unit, satisfying the equation $i^2 = -1$.
Real part:

$$Re(a + bi) := a$$

Imaginary part:

$$Im(a + bi) := b$$

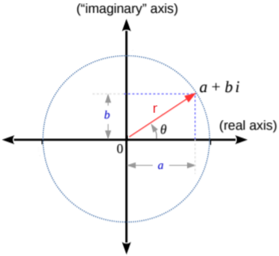
Complex conjugate:

$$\overline{a + bi} := a - bi$$

That is, negate the imaginary component.

Complex numbers

The complex plane is a geometric representation of the complex numbers established by the real axis and the perpendicular imaginary axis.



Here r is the absolute value (or modulus or magnitude) of the complex number z

$$r = \sqrt{a^2 + b^2}$$

and θ the argument of z

$$a = rcos\theta \qquad \beta = rsin\theta$$

Discriminant $\Delta<0$

Solve $y_{t+2} + 2y_{t+1} + 2y_t = 2^t$ assuming arbitrary initial conditions. Firstly consider the HE:

$$y_{t+2} + 2y_{t+1} + 2y_t = 0$$

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

$$a_1 = 2, \quad a_2 = 2$$

while the necessary and sufficient conditions for stability, namely:

$$1 + a_1 + a_2 > 0$$

$$1 - a_2 > 0$$

$$1 - a_1 + a_2 > 0$$

are not simultaneously satisfied.

Discriminant Δ<0

Moreover:

$$\lambda^2 + 2\lambda + 2y_t = 0$$

$$\Delta = (2)^2 - 4 \cdot 2 = -4 < 0$$

Since $\Delta < 0$, the roots of the characteristic equation are a complex conjugates:

$$\lambda_{1,2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

Cartesian coordinates:

$$(a, b) = (-1, 1)$$

The absolute value (or modulus or magnitude) of the complex number $z = -1 + i$ is:

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Discriminant $\Delta<0$

The argument of the complex number, denoted $arg(z)$ and labeled θ:

$$a = rcos\theta \Rightarrow -1 = \sqrt{2}cos\theta \Rightarrow cos\theta = \frac{-1}{\sqrt{2}}$$

$$b = rsin\theta \Rightarrow 1 = \sqrt{2}sin\theta \Rightarrow sin\theta = \frac{1}{\sqrt{2}}$$

From the trigonometric tables we find that the angle whose sine is $\frac{1}{\sqrt{2}}$ and whose cosine is $\frac{-1}{\sqrt{2}}$, is $\frac{3\pi}{4}$.
Therefore, the solution of the homogeneous equation is:

$$y_t = r^t(A_1 cos\theta t + A_2 sin\theta t) \Rightarrow y_t = \sqrt{2}^t(A_1 cos(\frac{3\pi}{4}t) + A_2 sin(\frac{3\pi}{4}t))$$

Discriminant $\Delta < 0$

Next consider the non-homogeneous equation:

$$y_{t+2} + 2y_{t+1} + 2y_t = 2^t$$

Try out an exponential guess function:

$$y_t = a2^t, \;\; y_{t+1} = a2^{t+1}, \;\; y_{t+2} = a2^{t+2}$$

Substituting in the non-homogeneous equation:

$$a2^{t+2} + 2a2^{t+1} + 2a2^t = 2^t \Leftrightarrow (a2^2 + 2a2 + 2a)2^t = 2^t \Leftrightarrow 10a2^t = 2^t$$

$$10a = 1 \Leftrightarrow a = \frac{1}{10}$$

Therefore:

$$PS: \;\; \bar{y}_t = \frac{1}{10}2^t$$

$$GS: \;\; y_t = A_1\sqrt{2}^t cos(\frac{3\pi}{4}t) + A_2\sqrt{2}^t sin(\frac{3\pi}{4}t) + \frac{1}{10}2^t$$

Operators

✓Operator: A mapping of one set into another, each of which has a certain structure.

✓For most practical purposes, operators can be treated just as algebraic quantities.

We introduce the lag operator L such that:

$$Ly_t = y_{t-1}$$

So,

$$L^2 y_t = y_{t-2},\ L^n y_t = y_{t-n},\ L^{-1} y_t = y_{t+1}$$

Formally, the operator L^n maps one sequence into another sequence.

Lag operator applied to a constant c

$$Lc = c$$

Distributive

$$(L^j + L^i) y_t = y_{t-j} + y_{t-i}$$

Associative

$$L^j L^i y_t = L^i L^j y_t = L^{i+j} y_t = y_{t-i-j}$$

Infinite-series expansions

Using a well-known infinite-series expansion , for $|c| < 1$, we have

$$\frac{1}{1-c} = 1 + c + c^2 + ... = \sum_{i=0}^{\infty} c^i$$

Treating aL exactly like c we have that,

$$\frac{1}{(1-aL)} = (1-aL)^{-1} = 1 + aL + a^2L^2 + ... = \sum_{i=0}^{\infty} a^i L^i$$

Notice that the above sequence is a bounded sequence if $|a| < 1$, but will divergent if $|a| > 1$. In this latter case, consider an alternative expansion:

$$(1-aL)^{-1} = -\sum_{i=1}^{\infty} (\frac{1}{a})^i L^{-i}$$

First-order Difference Equations

The general form is:

$$b_1 y_t + b_0 y_{t-1} = g(t)$$

✓It may happen that we do not know the functional form of g(t).
✓We know the actual succession of values $g(0), g(1), ..., g(t)$. In other words, $g(t)$ is a sequence of known real values
✓In such case the method of undetermined coefficients cannot be applied.
✓It is possible to find a particular solution by applying operational methods.

Backward and forward solutions

Solve:

$$b_1 y_t + b_0 y_{t-1} = x(t)$$

where $x(t)$ is a sequence of known real values.

$$y_t + \frac{b_0}{b_1} y_{t-1} = \frac{x(t)}{b_1}$$

Setting $b = \frac{b_0}{b_1}$ and $X_t = \frac{x(t)}{b_1}$ we have:

$$y_t + b y_{t-1} = X_t$$

The solution of the homogeneous equation is:

$$y_t^h = c(-b)^t$$

where c is an arbitrary constant.

Backward and forward solutions

Next, find a particular solution:

$$y_t + by_{t-1} = X_t$$

$$y_t + bLy_t = X_t$$

$$(1 + bL)y_t = X_t$$

If $|-b| < 1$, then the particular solution is:

$$\bar{y}_t = (1 - (-b)L)^{-1}X_t = \sum_{i=0}^{\infty}(-b)^i L^i X_t = \sum_{i=0}^{\infty}(-b)^i X_{t-i}$$

Note that $\bar{y}_t$ is a bounded sequence if $|-b| < 1$ (same: $|b| < 1$) but will be divergent if $|-b| > 1$. In this latter case consider an alternative expansion.

$$\bar{y}_t = (1 - (-b)L)^{-1}X_t = -\sum_{i=1}^{\infty}(-\frac{1}{b})^i L^{-i} X_t = -\sum_{i=1}^{\infty}(-\frac{1}{b})^i X_{t+i}$$