MSc MATHS ECON - Tutorial 1

Optimal Control & Discrete-time dynamic programming applications

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Optimal Control -The Typical Problem

$$\max_{u(t)} \int_{t_0}^{t_1} e^{-\rho t} F(t, x(t), u(t)) dt$$

$$s.t. \quad \frac{dx(t)}{dt} \equiv \dot{x}(t) = g(x(t), u(t), t)$$

$$x(t_0) = x_0$$

$$x(t_1) = x_1 \, fixed, \quad or \quad x(t_1) \geq x_1, \quad or \quad x(t_1) \quad free$$

$x(t)$: state variable, $u(t)$: control variable, g: equation of motion or transition equation, $[t_0, t_1]$ finite time horizon.

Finite-horizon with fixed endpoint

Solve the following optimal control problem:

$$\max_{\{u_t\}} \int_0^1 (y(t) - u(t)^2)dt$$

$$s.t. \quad \dot{y}(t) = u(t) \quad y(0) = 2 \quad y(1) = a$$

Finite-horizon with fixed endpoint

The Hamiltonian is:

$$H(t, y, u) = y(t) - u(t)^2 + \lambda(t)u(t)$$

According to Maximum Principle:

$$\frac{\partial H(t)}{\partial u(t)} = 0 \Rightarrow -2u(t) + \lambda(t) = 0 \Leftrightarrow u(t) = \frac{\lambda(t)}{2} \tag{1}$$

The Maximum Principle Conditions are:

$$\dot{\lambda}(t) = -\frac{\partial H(t)}{\partial y(t)} \Leftrightarrow \dot{\lambda}(t) = -1 \Leftrightarrow \lambda(t) = -t + c_1 \tag{2}$$

and

$$\dot{y}(t) = u(t) \tag{3}$$

Using (1), (2) and (3) we have that:

$$\dot{y}(t) = \frac{-t + c_1}{2}$$

Direct integration of the last equation yields:

$$y(t) = \frac{-t^2}{4} + \frac{c_1}{2}t + c_2$$

Finite-horizon with fixed endpoint

Use the initial and terminal values in order to determine arbitrary constants c_1 and c_2.

$$t = 0: \quad c_2 = 2$$

$$t = 1: \quad a = -\frac{1}{4} + \frac{c_1}{2} + c_2 \Leftrightarrow c_1 = 2a - \frac{7}{2}$$

Hence, the optimal paths for this problem are:

$$y^*(t) = \frac{-t^2}{4} + (\alpha - \frac{7}{4})t + 2$$

$$\lambda^*(t) = -t + 2a - \frac{7}{2}$$

$$u^*(t) = -\frac{t}{2} + a - \frac{7}{4}$$

Finite-horizon, free-terminal state

Solve the following optimal control problem

$$\max_{\{u_t\}} \int_0^1 (y(t) - u(t)^2)dt$$

$$s.t. \quad \dot{y}(t) = u(t) \quad y(0) = 5 \quad y(1) \; free$$

The same problem except that the terminal state $y(1)$ is left unrestricted.

$$y(t) = \frac{-t^2}{4} + \frac{c_1}{2}t + c_2, \quad u(t) = \frac{\lambda(t)}{2}, \quad \lambda(t) = -t + c_1$$

Condition $\lambda(T) = 0$ is the transversality condition appropriate for the free-terminal state problem. Hence,

$$\lambda(1) = 0 : \quad 0 = -1 + c_1 \Leftrightarrow c_1 = 1$$

$$t = 0 : \quad 5 = c_2$$

Finite-horizon, free-terminal state

So, the optimal paths for this problem are:

$$y^*(t) = \frac{-t^2}{4} + \frac{t}{2} + 5$$

$$\lambda^*(t) = -t + 1$$

$$u^*(t) = \frac{-t + 1}{2}$$

Finite-horizon with scrap value and discounted objective

Solve the following optimal control problem:

$$\max_{\{c(t)\}} \{ \int_0^T e^{-\theta t}(c(t) - 0.5c(t)^2)dt \} + e^{-\theta T} ak(T)$$

$$s.t. \quad \dot{k}(t) = k(t) - c(t)$$

$$k(0) = 1 > a > 0$$

$$\varphi(k(T)) = ak(T)$$

and $0 < \theta < 1$ is the rate of time preference and φ is the terminal curve.

Finite-horizon with scrap value and discounted objective

Use the current value Hamiltonian:

$$H^{cv}(t, c(t), k(t)) = c(t) - 0.5c(t)^2 + \mu(t)(k(t) - c(t))$$

According to Maximum Principle:

$$\frac{\partial H^{cv}}{\partial c(t)} = 0 \Rightarrow 1 - c(t) - \mu(t) = 0 \Leftrightarrow c(t) = 1 - \mu(t) \tag{4}$$

The Maximum Principle Conditions are:

$$\dot{\mu}(t) = -\frac{\partial H^{cv}}{\partial k(t)} + \theta\mu(t) \Rightarrow$$

$$\dot{\mu}(t) = -\mu(t) + \theta\mu(t) \tag{5}$$

and

$$\dot{k}(t) = k(t) - c(t) \tag{6}$$

From (4) and (6) we have that:

$$\dot{k}(t) = k(t) - 1 + \mu(t) \tag{7}$$

Hence, we end up with the following linear differential system:

$$\begin{Bmatrix} \dot{k}(t) = k(t) - 1 + \mu(t) \\ \dot{\mu}(t) = \mu(t)(-1+\theta) \end{Bmatrix}$$

Alternatively, we can solve (5):

$$\dot{\mu}(t) = -\mu(t) + \theta\mu(t) \Leftrightarrow \frac{\dot{\mu}(t)}{\mu(t)} = -1 + \theta$$

Direct integration yields:

$$ln\mu(t) = (-1+\theta)t + c \Leftrightarrow \mu(t) = e^{(-1+\theta)t+c} \Leftrightarrow \mu(t) = e^{(-1+\theta)t}e^{c}$$

Setting $e^{c} \equiv c_1$ we get:

$$\mu^*(t) = c_1 e^{(-1+\theta)t} \tag{8}$$

From (7) and (8) we have that:

$$\dot{k}(t) = k(t) - 1 + c_1 e^{(-1+\theta)t}$$

The solution of the above first order ode is:

$$k^*(t) = c_2 e^t - (\frac{c_1}{2-\theta})e^{(-1+\theta)t} + 1 \tag{9}$$

Use the transversality condition:

$$\mu(T) = \varphi'(k(T)) \Rightarrow \mu(T) = a$$

Substituting for $\mu^*(t)$ into (8) yields:

$$a = c_1 e^{(-1+\theta)T} \Leftrightarrow c_1 = ae^{(1-\theta)T}$$

Moreover, from the initial boundary and (9) we have that:

$$k(0) = 1 \Rightarrow 1 = c_2 e^0 - (\frac{c_1}{2-\theta})e^{(-1+\theta)0} + 1 \Leftrightarrow c_2 = \frac{c_1}{2-\theta}$$

Hence, the solution set:

$$\left\{ \begin{array}{c} \mu^*(t) = ae^{(1-\theta)(T-t)} \\ k^*(t) = \frac{ae^{(1-\theta)}}{2-\theta}(e^{(T+t)} - e^{(T-t)}) + 1 \\ c^*(t) = 1 - \mu^*(t) \end{array} \right\}$$

The Ramsey optimal growth model

Solve the following optimal growth problem. All quantities are in per capita terms. Utility is concave and the production function is classical CRS.

$$\max_{\{c_t\}} \int_0^\infty e^{-\theta t} u(c(t)) dt$$

$$s.t. \quad \dot{k}(t) = f(k(t)) - c(t) - nk(t)$$

$$k(0) = k_0$$

n is the rate of growth of population

Use the current-value formulation:

$$H^{cv}(t, c(t), k(t)) = u(c(t)) + \mu(t)(f(k(t)) - c(t) - nk(t))$$

According to Maximum Principle:

$$\frac{\partial H^{cv}}{\partial c(t)} = 0 \Rightarrow u'(c(t)) = \mu(t) \Rightarrow \tag{10}$$

$$\dot{\mu}(t) = u''(c(t))\dot{c}(t) \tag{11}$$

From the Maximum Principle Condition:

$$\dot{\mu}(t) = -\frac{\partial H^{cv}}{\partial k(t)} + \theta\mu(t) \Rightarrow \dot{\mu}(t) = -\mu(t)(f'(k(t)) - n) + \theta\mu(t) \tag{12}$$

Plugging (10) into (12) yields the Euler equation:

$$u''(c(t))\dot{c}(t) = -\mu(t)(f'(k(t)) - n) + \theta\mu(t) \overset{10}{\Longrightarrow}$$

The Ramsey optimal growth model

$$u''(c(t))\dot{c}(t) = -u'(c(t))(f'(k(t)) - n + \theta) \Longleftrightarrow$$

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'(c(t))}{u''(c(t))}\frac{1}{c(t)}(f'(k(t)) - n + \theta)$$

Moreover,

$$\dot{k}(t) = f(k(t)) - c(t) - nk(t)$$

and

$$\lim_{t \to \infty} \mu(t)e^{-\theta t}k^*(t) = 0$$

or using (10)

$$\lim_{t \to \infty} u'(c(t))e^{-\theta t}k^*(t) = 0$$

which implies that it would not be optimal to end up with positive capital because it could be consumed instead, since marginal utility of consumption, $u'(c(t))$, and its present value $u'(c(t))e^{-\theta t}$, is positive by assumption (concave utility function). Hence, we forced, terminal per capita capital to be zero.

Dynamic Programming-The typical problem

Discrete time dynamic optimization problems can be solved using Lagrange multiplier methods or dynamic programming methods. We will focus on dynamic programming methods. Consider the infinite optimal control problem:

$$\max_{\{v_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, v_t)$$

$$s.t. \quad x_{t+1} = g(x_t, v_t) \quad x_0 = \bar{x}_0$$

$\beta \epsilon (0,1)$ is the discount factor. We assume that $F(x_t, v_t)$ is a concave function and the set $\{(x_{t+1}, x_t) : x_{t+1} \leq g(x_t, v_t), v_t \epsilon \mathbb{R}^m\}$ is convex and compact. We are seeking a time invariant feedback rule or police function $v_t = h(x_t)$ such that the sequence $\{v_t\}_{t=0}^{\infty}$ generated by:

$$v(t) = h(x_t)$$

$$x_{t+1} = g(x_t, v_t) \quad x_0 = \bar{x}_0$$

solve the dynamic programming problem.

Methods for solving the dynamic programming problem

1. First order conditions and envelope condition
2. Undetermined coefficients (Guess and verify the value function)
3. Iterations

First order conditions and envelope condition

Solve the following optimal growth problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad k_{t+1} = f(k_t) - c_t$$

The Bellman equation for the problem is:

$$V(k_t) = \max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \{u(c_t) + \beta V(k_{t+1})\} \tag{13}$$

Substituting for $c_t = f(k_t) - k_{t+1}$ into the Bellman equation (13) yields:

$$V(k_t) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \{u(f(k_t) - k_{t+1}) + \beta V(k_{t+1})\}$$

Find the FOC:

$$\frac{\partial V}{\partial k_{t+1}} = 0 \Rightarrow u'(f(k_t) - k_{t+1})(-1) + \beta V'(k_{t+1}) = 0 \Leftrightarrow$$

$$u'(f(k_t) - k_{t+1}) = \beta V'(k_{t+1}) \tag{14}$$

First order conditions and envelope condition

Use the envelope condition:

$$V'(k_t) = u'(f(k_t) - k_{t+1})f'(k_t)$$

$$t+1: \quad V'(k_{t+1}) = u'(c_{t+1})f'(k_{t+1}) \tag{15}$$

Plugging (15) into (14) yields the Euler equation which holds for every t:

$$u'(c_t) = \beta u'(c_{t+1})f'(k_{t+1})$$

So, we derived the following difference system in c, k:

$$\begin{aligned} k_{t+1} &= f(k_t) - c_t && \textit{resource constraint} \\ u'(c_t) &= \beta u'(c_{t+1})f'(k_{t+1}) && \textit{Euler} \end{aligned}$$

Substituting the resource constraint into the Euler yields a second-order difference equation:

$$u'(f(k_t) - k_{t+1}) = \beta u'(f(k_{t+1}) - k_{t+2})f'(k_{t+1})$$

The Cake Eating Problem

Suppose an agent faces the following problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$s.t. \quad w_{t+1} = w_t - c_t$$

where w_t is the size of a cake at time t

First order conditions and envelope condition

The Bellman equation for the problem is:

$$V(w_t) = \max_{\{c_t, w_{t+1}\}_{t=0}^{\infty}} \{u(c_t) + \beta V(w_{t+1})\}$$

Substituting for $c_t = w_t - w_{t+1}$ into the value function yields:

$$V(w_t) = \max_{\{w_{t+1}\}_{t=0}^{\infty}} \{u(w_t - w_{t+1}) + \beta V(w_{t+1})\}$$

Find the FOC:

$$\frac{\partial V}{\partial w_{t+1}} = 0 \Rightarrow u'(w_t - w_{t+1})(-1) + \beta V'(w_{t+1}) = 0 \Leftrightarrow$$

$$u'(w_t - w_{t+1}) = \beta V'(w_{t+1}) \tag{16}$$

First order conditions and envelope condition

Use the envelope condition:

$$V'(w_t) = u'(w_t - w_{t+1})$$

$$t+1: \quad V'(w_{t+1}) = u'(c_{t+1}) \tag{17}$$

Plugging (17) into (16) yields the Euler equation which holds for every t:

$$u'(c_t) = \beta u'(c_{t+1})$$

So, we derived the following difference system in c, w:

$$\begin{aligned} w_{t+1} = w_t - c_t \qquad & \textit{constraint} \\ u'(c_t) = \beta u'(c_{t+1}) \qquad & \textit{Euler} \end{aligned}$$

Undetermined coefficients (Guess and verify the value function)

Guess and verify only works for a small subset of problems, where the functional form of the objective function is known

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t lnc_t$$

$$s.t. \quad w_{t+1} = w_t - c_t$$

The Bellman equation for the problem is:

$$V(w_t) = \max_{\{c_t, w_{t+1}\}_{t=0}^{\infty}} \{lnc_t + \beta V(w_{t+1})\}$$

Guess a form for the value function $V^G(w) = E + Fln(w)$ and substitute the guess into the Bellman equation:

$$E + Fln(w_t) = \max_{\{w_{t+1}\}_{t=0}^{\infty}} \{ln(w_t - w_{t+1}) + \beta(E + Fln(w_{t+1}))\}$$

Undetermined coefficients (Guess and verify the value function)

Perform the optimization on the RHS and obtain the policy function:

$$\frac{\partial V}{\partial w_{t+1}} = 0 \Rightarrow \frac{1}{w_t - w_{t+1}} = \frac{\beta F}{w_{t+1}} \Leftrightarrow w^*_{t+1} = \frac{\beta F w_t}{1+\beta F} \qquad (18)$$

Substitute the policy function into the Bellman equation and verify that the form of $V(w_t)$ is the same as $V(w_{t+1})$. In practice you try to obtain the coefficients of the guess value function.

$$E + F ln w_t = ln(w_t - \frac{\beta F w_t}{1+\beta F}) + \beta(E + F ln(\frac{\beta F w_t}{1+\beta F})) \Leftrightarrow$$

$$E + F ln w_t = ln(\frac{w_t}{1+\beta F}) + \beta E + \beta F ln(\frac{\beta F w_t}{1+\beta F}) \Leftrightarrow$$

$$E + F ln w_t = ln(w_t) - ln(1+\beta F) + \beta E + \beta F ln(\beta F) + \beta F ln(w_t) - \beta F ln(1+\beta F)$$

$$E + F ln w_t = \beta E + \beta F ln(\beta F) - (1+\beta F) ln(1+\beta F) + (1+\beta F) ln(w_t)$$

Undetermined coefficients (Guess and verify the value function)

Equating coefficients on both sides yields a value for F:

$$F = (1 + \beta F) \Leftrightarrow F = \frac{1}{1-\beta} > 0$$

Substituting for F and solving for E yields

$$E = \beta E + \frac{\beta}{1-\beta} ln(\frac{\beta}{1-\beta}) - (1 + \frac{\beta}{1-\beta}) ln(1 + \frac{\beta}{1-\beta}) \Leftrightarrow$$

$$E = \frac{1}{1-\beta}(ln(1-\beta) + \frac{\beta}{1-\beta} ln(\beta))$$

Undetermined coefficients (Guess and verify the value function)

Use the coefficients to identify the policy function. Using (18) we have that

$$w_{t+1}^* = \frac{\beta(\frac{1}{1-\beta})w_t}{1+\beta(\frac{1}{1-\beta})} \Leftrightarrow$$

$$w_{t+1}^* = \beta w_t$$

and using the transition equation $w_{t+1} = w_t - c_t$ we have

$$\beta w_t = w_t - c_t \Rightarrow$$

$$c_t^* = (1-\beta)w_t$$