TECHNIQUES OF DYNAMIC OPTIMIZATION IN CONTINUOUS AND DISCRETE TIME MODELS (GENERAL SPECIFICATION)

References:
[1] Kamien, M.I., and N.L. Schwartz (1991): *"Dynamic Optimization"*. North Holland Elsevier. Second Edition. (Cited "K+S").
[2] Sargent, Thomas J. (1987): *"Macroeconomic Theory"*. Academic Press. (Cited "Sargent, 1987a").
[3] Chapter 1 - "Dynamic Programming", in Thomas J. Sargent (1987), *Dynamic Macroeconomic Theory*, 11-56. Cambridge: Harvard University Press. Second Edition. (Cited "Sargent, 1987b, Ch. 1").

Notation:
Let a function of two variables $\phi[u(t), v(t)]$.
Its time derivative will be denoted ϕ' where $\phi' \equiv \frac{d\phi}{dt} \equiv \dot{\phi}$.
Its partial derivatives will be denoted ϕ_u, ϕ_v where $\phi_u \equiv \frac{\partial \phi}{\partial u}, \phi_v \equiv \frac{\partial \phi}{\partial v}$.
To simplify notation, time-varying variables $u(t), v(t)$ will be denoted u, v in continuous-time and u_t, v_t in discrete-time problems.

CONTINUOUS-TIME PROBLEMS

Suppose (today at time zero) we want to choose paths $\{x, u\}_0^\infty$ so as to solve the following (infinite-horizon) optimization problem:

$$max \int_0^\infty e^{-rt} f(x, u) dt \tag{1}$$

subject to

$$x' = g(x, u) \tag{2}$$

given the initial condition $x(0) = x_0$,

where,
r is the continuous-time *discount rate*,
e^{-rt} is the continuous-time *discount factor*,
x is the *state* variable, and
u is the *control* variable.

(2) is the called the *state* or *transition* equation.
Both x and u (and the associated functions f and g) are *time-varying*.

This problem may be solved in any of three ways:

- calculus of variations
- optimal control (the Hamiltonian equation)
- dynamic programming (the Bellman equation).

I. OPTIMAL CONTROL (THE HAMILTONIAN EQUATION)

Define the *present-value* Hamiltonian

$$H \equiv e^{-rt} f(x,u) + \lambda g(x,u) \tag{3}$$

or, the *current-value* Hamiltonian (see K+S, pp. 164-6)

$$e^{rt} H \equiv \mathcal{H} = f(x,u) + e^{rt} \lambda g(x,u) \tag{4}$$

letting $e^{rt}\lambda \equiv m$ so that (4) becomes

$$\mathcal{H}(x,u,m) = f(x,u) + m g(x,u) \tag{5}$$

where λ and m are *time-varying* multipliers associated with constraint (2).

Specifically, m_t is the *marginal value* of state variable x at time t while λ_t is the *marginal value* of state variable x at time t *discounted* back to time zero where the problem is being solved.

First-order conditions (FOCs):

$$\mathcal{H}_u = f_u(x,u) + m g_u(x,u) = 0 \tag{6}$$

$$m' \left(\equiv \frac{dm}{dt} \equiv \dot{m} \right) = rm - \mathcal{H}_x = rm - f_x(x,u) - m g_x(x,u) \tag{7}$$

$$\mathcal{H}_m = g(x,u) = x' \left(\equiv \frac{dx}{dt} \equiv \dot{x} \right) \tag{8}$$

Thus, the FOCs make up a system of three equations in three unknowns, namely (x, u, m).

Condition (7) is derived as follows (see K+S, p. 165):

We defined $m \equiv e^{rt}\lambda$.
Taking the time derivative of this expression, we obtain $m' = re^{rt}\lambda + e^{rt}\lambda' = m\lambda + e^{rt}\lambda'$.
At the optimum $\lambda' + H_x = 0$ ie the change in the marginal value of the state variable, λ', plus the marginal utility of the state variable, H_x, must equal zero (see K+S, pp. 136-141).
Thus $\lambda' = -H_x$.
Using this, we may write: $m' = rm - e^{rt} H_x \overset{\text{since } H \equiv e^{-rt}\mathcal{H}}{\Leftrightarrow} m' = rm - e^{rt} e^{-rt} \mathcal{H}_x \Leftrightarrow m' = rm - \mathcal{H}_x$. QED

II. DYNAMIC PROGRAMMING (THE BELLMAN EQUATION)

(See K+S, pp. 259-263).

Define the optimal value function $J(t_0, x_0)$ as the best value function that can be obtained starting at time t_0 in state x_0 where $0 \leq t_0 \leq \infty$:

$$J(t_0, x_0) \equiv max \int_{t_0}^{\infty} e^{-rt} f(t, x, u) dt \tag{9a}$$

subject to $x' = g(x, u)$
given the initial condition $x(0) = x_0$,

or, equivalently:

$$J(t_0, x_0) \equiv max \left\{ e^{-rt_0} \int_{t_0}^{\infty} e^{-r(t-t_0)} f(x, u) dt \right\} \tag{9b}$$

Defining

$$V(x_0) \equiv max \int_{t_0}^{\infty} e^{-r(t-t_0)} f(x, u) dt \tag{10}$$

as the value of the integral, we may write the optimal value function $J(t_0, x_0)$, as follows:

$$J(t_0, x_0) \equiv max \left\{ e^{-rt_0} V(x_0) \right\} \tag{11}$$

Note that the value, $V(x_0)$, of the integral on the RHS of equation $(9b)$, $\int_{t_0}^{\infty} e^{-r(t-t_0)} f(x, u) dt$, depends on the initial state, x_0, but is independent of the initial time, t_0, ie it only depends on elapsed time, $(t - t_0)$.

The partial derivatives of $J(t_0, x_0)$ are $J_{t_0}(t_0, x_0) = -re^{-rt_0} V(x_0)$ and $J_{x_0}(t_0, x_0) = e^{-rt_0} V'(x_0)$. Dropping the time subscripts, $J(t, x) = e^{-rt} V(x)$, $J_t(t, x) = -re^{-rt} V(x)$ and $J_x(t, x) = e^{-rt} V'(x)$.

We can prove (see K+S, p. 260) that

$$-J_t(t, x) \simeq \max_u \left[e^{-rt} f(x, u) + J_x(t, x) g(x, u) \right] \tag{12}$$

or, substituting for the partial derivatives

$$re^{-rt} V(x) \simeq \max_u \left[e^{-rt} f(x, u) + e^{-rt} V'(x) g(x, u) \right] \tag{13}$$

and multiplying both sides by e^{rt}, we obtain the *Hamilton-Jacobi-Bellman equation*

$$rV(x) \simeq \max_u \left[f(x, u) + V'(x) g(x, u) \right] \tag{14}$$

which is the fundamental partial differential equation obeyed by the optimal value function $J(t, x)$ (see K+S, p. 261).

<u>FOCs:</u>

The first order conditions of the Hamilton-Jacobi-Bellman equation are:

$$0 = f_u(x,u) + V'(x) g_u(x,u) \tag{15}$$

$$rV'(x) = f_x(x,u) + V''(x) g(x,u) + V'(x) g_x(x,u) \tag{16}$$

where $V'(x)$ is the derivative of $V(x)$ with respect to x, $V'(x) \equiv \frac{dV(x)}{dx}$.

Equation (15) is the marginal condition resulting from differentation of the Hamilton-Jacobi-Bellman equation with respect to control variable u, while equation (16) is the so-called *envelope* or *Benveniste-Scheinkman condition* resulting from taking the derivative of the Hamilton-Jacobi-Bellman equation with respect to state x, after substituting for the optimal solution of control variable u, which we obtain by solving equation (15) together with the constraint $x' = g(x,u)$ for u. The optimal solution for u thus obtained is a function of x.

The conditions given by equations $(15) - (16)$ are similar to those given by equations $(6) - (7)$ in the optimal control solution above. This becomes obvious if we set $\lambda = J_x(t,x)$ and, given the fact that $J_x(t,x) = e^{-rt}V'(x)$, substitute for λ in the definition of the current-value multiplier, $e^{rt}\lambda \equiv m$, which yields $m = V'(x)$ ie if the current-value multiplier equals the marginal value of the state variable, x. Moreover, if $m = V'(x)$, then $\dot{m} = V''(x)\dot{x}$, where $\dot{x} = g(x,u)$ ie the optimization constraint. Thus, the dynamic programming solution coincides with the optimal control solution.

Given the above FOCs, we then work as usually by linearizing and studying the long-run and the transition (dynamic stability) (see K+S, pp. 175-7).

DISCRETE-TIME PROBLEMS

We will consider three solutions:

- dynamic programming (the Bellman equation)
- optimal control (the Hamiltonian equation)
- the Lagrangean equation.

I. DYNAMIC PROGRAMMING

(See Sargent, 1987b, Ch. 1).

The Bellman equation is:

$$V\left(x_t\right) = \max_{\{u_t, x_{t+1}\}} \left[f\left(x_t, u_t\right) + \delta V\left(x_{t+1}\right)\right] \tag{1}$$

subject to

$$x_{t+1} - x_t = g\left(x_t, u_t\right) \tag{2}$$

or, equivalently:

$$V\left(x_t\right) = \max_{u_t} \left\{f\left(x_t, u_t\right) + \delta V\left[x_t + g\left(x_t, u_t\right)\right]\right\} \tag{3}$$

where δ is the one-period discrete-time *discount factor*.

FOCs:

$$0 = f_u(.t) + \delta V'\left(x_{t+1}\right) g_u(.t) \tag{4}$$

$$V'\left(x_t\right) = f_x(.t) + \delta V'\left(x_{t+1}\right)\left[1 + g_x(.t)\right] \tag{5}$$

where equation (5) is the envelope condition for the state variable at time t, x_t.

We can use equation (4) to substitute out $V'\left(x_t\right)$ or $V'\left(x_{t+1}\right)$, so that equations (5) and (2) are a system of two equations in two unknowns, namely x_t, u_t.

Application

In the basic optimal growth model, we have:

$f\left(x_t, u_t\right) = \upsilon\left(c_t\right)$[1]
$g\left(x_t, u_t\right) = f\left(k_t\right) - c_t$ or $k_{t+1} - k_t = f\left(k_t\right) - c_t$

The Bellman equation is:
$V(k_t) = \max_{\{c_t, k_{t+1}\}}[\upsilon(c_t) + \delta V(k_{t+1})]$ s.t. $k_{t+1} - k_t = f\left(k_t\right) - c_t$
or, equivalently $V(k_t) = \max_{k_{t+1}}\{\upsilon[f(k_t) - k_{t+1} + k_t] + \delta V(k_{t+1})]$.

Then, equation (4) becomes $0 = \upsilon\prime[f(k_t) - k_{t+1}](-1) + \delta V\prime(k_{t+1}) \Leftrightarrow \upsilon\prime(c_t) = \delta V\prime(k_{t+1})$, and equation (5) becomes $V\prime(k_t) = \upsilon\prime(c_t)\left[f\prime(k_t) + 1\right] \overset{\upsilon\prime(c_t) = \delta V\prime(k_{t+1})}{\Leftrightarrow} V\prime(k_t) = \delta V\prime(k_{t+1})\left[1 + f\prime(k_t)\right]$ which is the envelope condition for k_t.

Now, using these two conditions we obtain the usual Euler equation:

$\left[\frac{\upsilon\prime(c_{t-1})}{\delta}\right] = \delta\left[\frac{\upsilon\prime(c_t)}{\delta}\right]\left[1 + f\prime(k_t)\right] \Leftrightarrow \upsilon\prime(c_t) = \delta\upsilon\prime(c_{t+1})\left[1 + f\prime(k_t)\right]$

where
$V\prime(k_{t+1}) = \frac{\upsilon\prime(c_t)}{\delta} = \upsilon\prime(c_{t+1}) \Rightarrow V\prime(k_t) = \frac{\upsilon\prime(c_{t-1})}{\delta} = \upsilon\prime(c_t)$.

[1] We use υ instead of u to denote the utility function in order to avoid confusion with the notation used for the control variable, u.

II. OPTIMAL CONTROL

Define the current-value Hamiltonian:

$$\mathcal{H}_t (x_t, u_t, m_t) \equiv f (x_t, u_t) + m_t g (x_t, u_t) \tag{6}$$

FOCs:

$$\mathcal{H}_u = 0 \Leftrightarrow f_u (.t) + m_t g_u (.t) = 0 \tag{7}$$

$$m_t - \delta \left(\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} \right) = \delta m_{t+1} \tag{8a}$$

Since $\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = f_x (.t+1) + m_{t+1} g_x (.t+1)$, equation $(8a)$ becomes

$$m_t = \delta \left\{ f_x (.t+1) + m_{t+1} \left[1 + g_x (.t+1) \right] \right\} \tag{8b}$$

$$\mathcal{H}_m = g (x_t, u_t) = x_{t+1} - x_t \tag{9}$$

The conditions given by equations $(7), (8b)$ and (9) are a system of three equations in three unknowns, namely x, u, m.

Equation $(8a)$ may be derived as follows:[2]

Since we are solving a discrete time problem, $\lambda_t = \delta^t m_t$ and $\lambda_{t+1} = \delta^{t+1} m_{t+1}$
(one period ahead) where $\delta^t = \frac{1}{(1+r)^t}$ is the t-period discount factor (starting at time 0).
Taking differences, $\lambda_{t+1} - \lambda_t = \delta^{t+1} m_{t+1} - \delta^t m_t \Leftrightarrow \lambda_{t+1} - \lambda_t = \delta^t (\delta m_{t+1} - m_t)$.
Similar to continuous time optimization, $\lambda_{t+1} - \lambda_t = -\frac{\partial H_{t+1}}{\partial x_{t+1}} = -\frac{\partial}{\partial x_{t+1}} \left(\delta^{t+1} \mathcal{H}_{t+1} \right)$.
Therefore, $\lambda_{t+1} - \lambda_t = -\frac{\partial}{\partial x_{t+1}} \left(\delta^{t+1} \mathcal{H}_{t+1} \right) = \delta^t (\delta m_{t+1} - m_t) \Leftrightarrow -\delta^t \delta \left(\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} \right) = \delta^t (\delta m_{t+1} - m_t) \Leftrightarrow$
$\Leftrightarrow -\delta \left(\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} \right) = \delta m_{t+1} - m_t \Leftrightarrow m_t - \delta \left(\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} \right) = \delta m_{t+1}$. QED

The optimal control solution coincides with the dynamic programming solution, if we let $m_t = \delta V' (x_{t+1})$.

[2] See also the derivation offered on page 10.

III. THE LAGRANGEAN EQUATION

"Sargent, 1987a" uses this method a lot.

Define the Lagrangean equation:

$$\mathcal{L} \equiv \max_{\{u_t, x_{t+1}\}} \sum_{t=0}^{\infty} \delta^t \left\{ f\left(x_t, u_t\right) + \mu_t \left[-x_{t+1} + x_t + g\left(x_t, u_t\right)\right]\right\}$$

where μ_t is the Lagrangean multiplier.

FOCs:

$$f_u\left(.t\right) + \mu_t g_u\left(.t\right) = 0$$

$$\delta f_x\left(.t+1\right) - \mu_t + \delta\mu_{t+1} + \delta\mu_{t+1} g_x\left(.t+1\right) = 0 \Longleftrightarrow \mu_t = \delta\left[f_x\left(.t+1\right) + \mu_{t+1} + \mu_{t+1} g_x\left(.t+1\right)\right] \Longleftrightarrow$$
$$\Longleftrightarrow \mu_t = \delta\left\{f_x\left(.t+1\right) + \mu_{t+1}\left[1 + g_x\left(.t+1\right)\right]\right\}$$

We conclude that this solution coincides with the optimal control and dynamic programming ones.

COMPARISON OF THE DYNAMIC EQUATIONS IN DISCRETE AND CONTINUOUS TIME

If δ is the discrete-time discount factor and r is the continuous-time discount factor, then $r = \frac{1-\delta}{\delta}$ given $\delta = \frac{1}{1+r}$, namely $\frac{1-\delta}{\delta} = \frac{1-\frac{1}{1+r}}{\frac{1}{1+r}} = r$. Hence equation (7) in the continuous-time model implies $\frac{1-\delta}{\delta}m - \frac{\partial \mathcal{H}}{\partial x} = m'$ or, in discrete-time, $\frac{1-\delta}{\delta}m_t - \frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = m_{t+1} - m_t \Leftrightarrow \frac{m_t}{\delta} - \frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = m_{t+1} \Leftrightarrow$ $\Longleftrightarrow m_t - \delta \frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = \delta m_{t+1}$ which is equation ($8a$) in the discrete-time model.

Note that both $0 < \delta < 1$ and $0 < r < 1$.
Moreover, δ is a *discount factor* while r is a *discount rate.*

THE BASIC OPTIMAL GROWTH MODEL IN DISCRETE TIME

Suppose we want to choose paths of $\{c_t, k_t\}_{t=0}^{\infty}$ so as to solve the following optimization problem:

$\max \sum_{t=0}^{\infty} \delta^t u(c_t)$ s.t. $k_t - k_{t-1} = f(k_{t-1}) - c_t$ given k_{-1} where $0 < \delta < 1$.[3]

We will solve the problem in two ways:

- dynamic programming
- optimal control.

[3] Alternatively, we could have used constraint $k_{t+1} - k_t = f(k_t) - c_{t+1}$ given k_0. This is the constraint used in the application on page 7.

I. DYNAMIC PROGRAMMING

Write the Bellman equation:

$$V\left(k_{t-1}\right)=\max_{\{c_t,k_t\}}\left[u\left(c_t\right)+\delta V\left(k_t\right)\right] \tag{1a}$$

or, equivalently:

$$V\left(k_{t-1}\right)=\max_{k_t}\left\{u\left[f\left(k_{t-1}\right)+k_{t-1}-k_t\right]+\delta V\left(k_t\right)\right\} \tag{1b}$$

FOC (wrt k_t):

$$0=u'\left(c_t\right)(-1)+\delta V'\left(k_t\right)\Longleftrightarrow u'\left(c_t\right)=\delta V'\left(k_t\right) \tag{2}$$

Envelope condition (wrt state k_{t-1}):

$V'\left(k_{t-1}\right)=u'\left(c_t\right)\left[f'\left(k_{t-1}\right)+1-\left(\frac{\partial k_t}{\partial k_{t-1}}\right)\right]+\delta V'\left(k_t\right)\left(\frac{\partial k_t}{\partial k_{t-1}}\right)\Leftrightarrow$
$\Leftrightarrow V'\left(k_{t-1}\right)=u'\left(c_t\right)\left[1+f'\left(k_{t-1}\right)\right]-\left[u'\left(c_t\right)-\delta V'\left(k_t\right)\right]\left(\frac{\partial k_t}{\partial k_{t-1}}\right).$
But $\left[u'\left(c_t\right)-\delta V'\left(k_t\right)\right]=0$ by the FOC. Hence: $V'\left(k_{t-1}\right)=u'\left(c_t\right)\left[1+f'\left(k_{t-1}\right)\right]$.
or, shifted one period forward:

$$V'\left(k_t\right)=u'\left(c_{t+1}\right)\left[1+f'\left(k_t\right)\right] \tag{3}$$

Plugging (3) into (2), we obtain the Euler equation:

$$u'\left(c_t\right)=\delta u'\left(c_{t+1}\right)\left[1+f'\left(k_t\right)\right] \tag{4}$$

II. OPTIMAL CONTROL

Define the *current-value* Hamiltonian:

$$\mathcal{H}_t \equiv u\left(c_t\right) + m_t\left[f\left(k_{t-1}\right) - c_t\right] \tag{5}$$

where $k_t - k_{t-1} = f\left(k_{t-1}\right) - c_t$.

FOCs:

$$\frac{\partial \mathcal{H}_t}{\partial c_t} = 0 \Leftrightarrow u'\left(c_t\right) = m_t \tag{6}$$

$$m_t - \delta\left(\frac{\partial \mathcal{H}_{t+1}}{\partial k_t}\right) = \delta m_{t+1} \tag{7a}$$

Since $\frac{\partial \mathcal{H}_{t+1}}{\partial k_t} = m_{t+1} f'\left(k_t\right)$, equation $(7a)$ becomes $m_t - \delta\left[m_{t+1} f'\left(k_t\right)\right] = \delta m_{t+1}$, or:

$$m_t = \delta m_{t+1}\left[1 + f'\left(k_t\right)\right] \tag{7b}$$

$$\frac{\partial \mathcal{H}_t}{\partial m_t} = f\left(k_{t-1}\right) - c_t = k_t - k_{t-1}$$

Plugging equations (6) into equation $(7b)$, we obtain the Euler equation:

$$u'\left(c_t\right) = \delta u'\left(c_{t+1}\right)\left[1 + f'\left(k_t\right)\right] \tag{8}$$

which is the same as dynamic programming equation (4).