LECTURE 12

Lecture 11

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$$y'(t) = \alpha_{11}\cdot y(t) + \alpha_{12}\cdot z(t) + g_1(t)$$
$$z'(t) = \alpha_{21}\, y(t) + \alpha_{22}\cdot z(t) + g_2(t)$$

$$\Leftrightarrow \begin{bmatrix} y' \\ z' \end{bmatrix} = \boxed{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} g_1(t) \\ g_2(t) \end{bmatrix}$$

α_{12} and/or α_{21} $\neq 0$

Homogeneous System

$$y' = \alpha_{11}\cdot y + \alpha_{12}\cdot z \quad (1) \rightarrow z = \frac{1}{\alpha_{12}} y' - \frac{\alpha_{11}}{\alpha_{12}} y$$
$$z' = \alpha_{21}\cdot y + \alpha_{22}\cdot z \quad (2)$$
$$z' = \frac{1}{\alpha_{12}}\cdot y'' - \frac{\alpha_{11}}{\alpha_{12}}\cdot y'$$

Solution

λ_1, λ_2 distinct real roots

$$y = A_1\cdot e^{\lambda_1\cdot t} + A_2\cdot e^{\lambda_2\cdot t} \Leftrightarrow$$
$$\Leftarrow \; y' = A_1\cdot\lambda_1\cdot e^{\lambda_1 t} + A_2\cdot\lambda_2\cdot e^{\lambda_2 t}$$

2nd Order

$$\frac{1}{\alpha_{12}}\cdot y'' - \frac{\alpha_{11}}{\alpha_{12}}\cdot y' = \frac{\alpha_{12}\,\alpha_{21}\cdot y}{\alpha_{12}} + \frac{\alpha_{22}}{\alpha_{12}}\cdot y' - \alpha_{22}\cdot\frac{\alpha_{11}}{\alpha_{12}}\cdot y \Leftrightarrow y'' - (\alpha_{11} + \alpha_{22})\cdot y' + (\alpha_{11}\cdot\alpha_{22} - \alpha_{12}\cdot\alpha_{21})\cdot y = 0$$

$$\lambda^2 - (\alpha_{11} + \alpha_{22})\cdot\lambda + (\alpha_{11}\cdot\alpha_{22} - \alpha_{12}\cdot\alpha_{21}) = 0$$

$$\lambda_1, \lambda_2$$

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$$\underset{n\times 1}{y'} = \underset{n\times n}{A}\cdot\underset{n\times 1}{y} + \underset{n\times 1}{g(t)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}_{n\times n}$$

$$y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

$$g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

n-th order

$$\alpha_0 y^{(n)} + \alpha_1 y^{(n-1)} + \dots + \alpha_{n-1} y' + \alpha_n y = G(t)$$

Define n-1 new auxiliary variables

$$\begin{aligned} y_1 &= y' \\ y_2 &= y'' = y_1' \\ y_3 &= y''' = y_2' \\ &\vdots \\ y_{n-1} &= y^{(n-1)} = y'_{n-2} \end{aligned}$$

$$y^{(n)} = -\frac{\alpha_1}{\alpha_0} y^{(n-1)} - \frac{\alpha_2}{\alpha_0} y^{(n-2)} - \dots - \frac{\alpha_{n-1}}{\alpha_0} y' - \frac{\alpha_n}{\alpha_0} y + \frac{G(t)}{\alpha_0}$$

$$\Leftrightarrow \quad y'_{n-1} = -\frac{\alpha_1}{\alpha_0} y_{n-1} - \frac{\alpha_2}{\alpha_0} y_{n-2} - \dots - \frac{\alpha_{n-1}}{\alpha_0} y_1 - \frac{\alpha_n}{\alpha_0} y + \frac{G(t)}{\alpha_0}$$

The corresponding system:

$$\begin{bmatrix} y' \\ y_1' \\ y_2' \\ \vdots \\ y'_{n-2} \\ y'_{n-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -\frac{\alpha_n}{\alpha_0} & -\frac{\alpha_{n-1}}{\alpha_0} & -\frac{\alpha_{n-2}}{\alpha_0} & \cdots & -\frac{\alpha_1}{\alpha_0} \end{bmatrix}}_{A} \cdot \begin{bmatrix} y \\ y_1 \\ y_2 \\ \vdots \\ y_{n-2} \\ y_{n-1} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \frac{g(t)}{\alpha_0} \end{bmatrix}}_{g(t)}$$

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$$\underset{n\times 1}{y'} = \underset{n\times n}{A} \cdot \underset{n\times 1}{y} + \underset{n\times 1}{g(t)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$$

$$y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$$

characteristic equation of A

$$\det(\underset{n\times n}{A} - \lambda \cdot \underset{n\times n}{I}) = 0$$

$$g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} = 0$$

n'th order polynomial

$\lambda_1, \lambda_2, \ldots, \lambda_n$

Homogeneous

$$y' = A \cdot y$$

Wise guess

$$y = \alpha \cdot e^{\lambda \cdot t}$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$y' = \alpha \cdot \lambda \cdot e^{\lambda \cdot t}$$

$$\Leftrightarrow \underset{n\times 1}{\alpha} \cdot \lambda \cdot e^{\lambda \cdot t} = \underset{n\times n}{A} \cdot \underset{n\times 1}{\alpha} \cdot e^{\lambda \cdot t} \Leftrightarrow$$

$$\Leftrightarrow A \cdot \alpha \cdot e^{\lambda \cdot t} - \lambda \cdot \alpha \cdot e^{\lambda \cdot t} = \underset{n\times 1}{0}$$

$$\Leftrightarrow \left[\underset{n\times n}{A} \cdot \underset{n\times 1}{\alpha} - \lambda \cdot \underset{n\times 1}{\alpha}\right] \cdot e^{\lambda \cdot t} = 0 \Leftrightarrow$$

$$\Leftrightarrow \left[\underset{n\times n}{A} - \lambda \cdot \underset{n\times n}{I}\right] \cdot \alpha \cdot e^{\lambda \cdot t} = 0 \Leftrightarrow$$

$$\Leftrightarrow e^{\lambda \cdot t} \cdot [A - \lambda I]\alpha = 0 \Leftrightarrow$$

$$\Leftrightarrow [A - \lambda I]\alpha = 0$$

$$\det(A - \lambda I) = 0$$

Roots $\rightarrow$ Eigenvalues of matrix A

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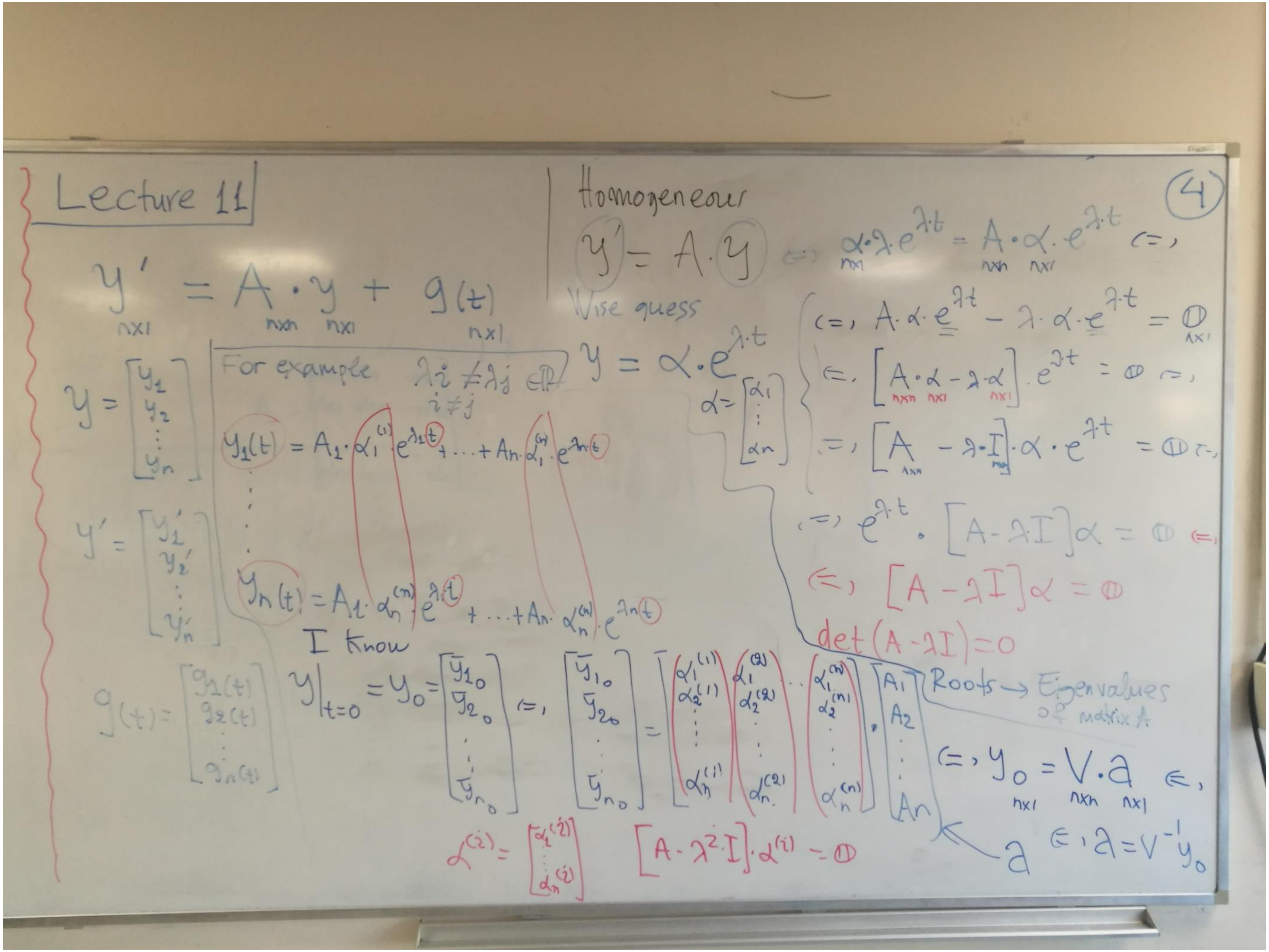

Lecture 11
$y'_{n\times1} = A_{n\times n}\cdot y_{n\times1} + g(t)_{n\times1}$
$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$
$y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}$
$g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$
For example $\lambda_i \neq \lambda_j \in \mathbb{R}$, $i \neq j$
$y_1(t) = A_1\cdot\alpha_1^{(1)} e^{\lambda_1 t} + \dots + A_n\cdot\alpha_1^{(n)}\cdot e^{\lambda_n t}$
$y_n(t) = A_1\cdot\alpha_n^{(n)} e^{\lambda_1 t} + \dots + A_n\cdot\alpha_n^{(n)}\cdot e^{\lambda_n t}$
I Know
$y\big|_{t=0} = y_0 = \begin{bmatrix} \bar{y}_{1_0} \\ \bar{y}_{2_0} \\ \vdots \\ \bar{y}_{n_0} \end{bmatrix} \Leftrightarrow \begin{bmatrix} \bar{y}_{1_0} \\ \bar{y}_{2_0} \\ \vdots \\ \bar{y}_{n_0} \end{bmatrix} = \begin{bmatrix} \alpha_1^{(1)} & \alpha_1^{(2)} & \dots & \alpha_1^{(n)} \\ \alpha_2^{(1)} & \alpha_2^{(2)} & & \alpha_2^{(n)} \\ \vdots & \vdots & & \vdots \\ \alpha_n^{(1)} & \alpha_n^{(2)} & & \alpha_n^{(n)} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$
$\alpha^{(i)} = \begin{bmatrix} \alpha_1^{(i)} \\ \vdots \\ \alpha_n^{(i)} \end{bmatrix}$ $[A - \lambda^i\cdot I]\cdot\alpha^{(i)} = \mathbb{O}$
Homogeneous
$y' = A\cdot y \Leftrightarrow \alpha_{n\times1}\cdot\lambda\cdot e^{\lambda t} = A_{n\times n}\cdot\alpha_{n\times1}\cdot e^{\lambda t} \Leftrightarrow$
Wise guess
$y = \alpha\cdot e^{\lambda t}$
$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$
$\Leftrightarrow A\cdot\alpha\cdot e^{\lambda t} - \lambda\cdot\alpha\cdot e^{\lambda t} = \mathbb{O}_{n\times1}$
$\Leftrightarrow [A_{n\times n}\cdot\alpha_{n\times1} - \lambda\cdot\alpha_{n\times1}]\cdot e^{\lambda t} = \mathbb{O} \Leftrightarrow$
$\Leftrightarrow [A_{n\times n} - \lambda\cdot I_{n\times n}]\cdot\alpha\cdot e^{\lambda t} = \mathbb{O} \Leftrightarrow$
$\Leftrightarrow e^{\lambda t}\cdot[A - \lambda I]\alpha = \mathbb{O} \Leftrightarrow$
$\Leftrightarrow [A - \lambda I]\alpha = \mathbb{O}$
$\det(A - \lambda I) = 0$
Roots → Eigenvalues of matrix A
$\Leftrightarrow y_{0\,n\times1} = V_{n\times n}\cdot a_{n\times1} \Leftrightarrow$
$a \Leftrightarrow a = V^{-1} y_0$
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$$\underset{n\times1}{y'} = \underset{n\times n}{A}\cdot\underset{n\times1}{y} + \underset{n\times1}{g(t)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} \qquad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Homogeneous

$$\underset{n\times1}{y'} = \underset{n\times n}{A}\cdot\underset{n\times1}{y}$$

Matrix exponential function : candidate solution (wise guess)

If $n=1$: $y' = \alpha\cdot y$

$$y = \gamma\cdot e^{\lambda\cdot t}$$

Discrete time

$$y = A^t\cdot c$$

$$e^{\lambda\cdot t} = 1 + \alpha\cdot t + \frac{\alpha^2}{2}\cdot t^2 + \frac{\alpha^3}{3!}\cdot t^3 + \dots + \frac{\alpha^n}{n!}\cdot t^n$$

$$\underset{n\times1}{y} = e^{\underset{n\times n}{A}\cdot t}\cdot\underset{n\times1}{c}$$

what is $e^{A\cdot t}$ → expand as a power series

→ Matrix power series expansion

$$e^{A\cdot t} = I + A\cdot t + A^2\cdot\frac{1}{2}\cdot t^2 + A^3\cdot\frac{1}{3!}\cdot t^3 + \dots + A^n\cdot\frac{1}{n!}\cdot t^n + \dots$$

I can define (take the derivative of 5)

$$\frac{d}{dt}e^{A\cdot t} = A + A^2\cdot t + A^3\cdot t^2\cdot\frac{1}{2} + \dots$$

$$= A\cdot\left[I + A\cdot t + \dots + A^n\cdot\frac{1}{n!}\cdot t^n + \dots\right] =$$

$$= A\cdot e^{A\cdot t}$$

$$y' = A\cdot y$$

$$A\cdot e^{A\cdot t}\cdot c = A\cdot e^{A\cdot t}\cdot c \checkmark$$

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$$\underset{n\times 1}{y'} = \underset{n\times n}{A} \cdot \underset{n\times 1}{y} + \underset{n\times 1}{g(t)}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \qquad y' = \begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} \qquad g(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Homogeneous

$$\underset{n\times 1}{y'} = \underset{n\times n}{A} \cdot \underset{n\times 1}{y}$$

Solution $y(t) = e^{A\cdot t} \cdot c$

$V^{-1} A \cdot V = \Lambda \Leftarrow$

$\Leftarrow A = V \cdot \Lambda \cdot V^{-1}$

$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) =$

$$= \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \; \lambda_n \end{bmatrix}$$

$\lambda_i \to$ eigenvalues of A

$y = e^{V\cdot\Lambda\cdot V^{-1} \cdot t} c$

If A can be diagonalized.

$A = V \cdot \Lambda \cdot V^{-1}$

$\hookrightarrow$ matrix of eigenvectors

matrix function $\varphi(A) = V \cdot \varphi(\Lambda) \cdot V^{-1}$

$\varphi(\Lambda) = \operatorname{diag}(\varphi(\lambda_1), \dots, \varphi(\lambda_n)) =$

$$= \begin{bmatrix} \varphi(\lambda_1) & & \emptyset \\ & \ddots & \\ \emptyset & & \varphi(\lambda_n) \end{bmatrix}$$

$$e^{A\cdot t} = V \cdot \begin{bmatrix} e^{\lambda_1 \cdot t} & & \emptyset \\ & e^{\lambda_2 \cdot t} & \\ \emptyset & & \ddots \; e^{\lambda_n \cdot t} \end{bmatrix} \cdot V^{-1}$$

Solution: $y(t) = V \cdot e^{\Lambda t} \cdot V^{-1} \cdot c =$

$= V \operatorname{diag}(e^{\lambda_1 t}, \dots, e^{\lambda_n t}) \cdot V^{-1} \cdot c$

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Back to Discrete time

Application

Rational Expectations

n endogenous variables $\begin{cases} m & \text{free (jump) (control)} \quad p_t \\ n-m & \text{state (predetermined)} \quad z_t \end{cases}$

r exogenous variables (policy, shocks) u_t

$p^e_{t+1,t} =:$ Rational Expectations

$= E(p_{t+1} \mid I_t)$

$I_t = \{ z_t, \{u^e_{t+j,t}\}_{j=0}^{\infty}, \theta \}$ Model parameters

Perfect foresight $\{u^e_{t+j,t}\}_{j=0}^{\infty} = \{u_{t+j}\}_{j=0}^{\infty}$

then $p^e_{t+1,t} = p_{t+1}$

Theory: $f(z_{t+1}, z_t, p^e_{t+1}, p_t, u_t; \theta) = 0$

Linear

Parameters

+ Perfect foresight: $p^e_{t+1} = p_{t+1}$ (7)

$$\begin{bmatrix} z_{t+1} \\ p^e_{t+1} \end{bmatrix} = A \cdot \begin{bmatrix} z_t \\ p_t \end{bmatrix} + B \cdot u_t$$