Lecture 11

1

Ordinary Differential Equations
Linear
(Continuous time)

$$y = f(t)$$

unknown function

$$y' \equiv \frac{dy}{dt}$$

$$y'' \equiv \frac{d^2y}{dt^2}$$

$$\vdots$$

$$y^{(n)} \equiv \frac{d^n y}{dt^n}$$

$$\frac{dy}{dt} = \alpha \Leftrightarrow \int dy = \alpha \int dt \Leftrightarrow y = \alpha \cdot t + c$$

$$\frac{d^2y}{dt^2} = \alpha \Leftrightarrow y = \frac{1}{2}\alpha \cdot t^2 + \underbrace{b \cdot t + C}_{\text{two arbitrary constants}}$$

{Perform two successive integrations}

General solution of the non-homogeneous

$$y(t) = f(t; A_1, A_2, \ldots, A_n) + \bar{y}(t)$$

Particular solution (in general depends on what $g(t)$ is)

Nth order linear Ordinary differential equation with constant coefficients

$$\alpha_0 \cdot y^{(n)} + \alpha_1 \cdot y^{(n-1)} + \ldots + \alpha_{n-1} \cdot y' + \alpha_n y = g(t)$$

$\alpha_0 \neq 0$

Homogeneous: $g(t) = 0$

↓ solution

$$f(t; A_1, \ldots, A_n) = A_1 \cdot y_1(t) + \ldots + A_n \cdot y_n(t)$$

If $y_1(t)$ a solution so is $A \cdot y_1(t)$

If $\underbrace{y_1(t), y_2(t)}_{\text{linearly independent}}$ → solutions, so is $A_1 \cdot y_1(t) + A_2 \cdot y_2(t)$

↓

Generalized for n functions

Wronski Determinant

$$W = \begin{vmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1'(t) & y_2'(t) & \ldots & y_n'(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{vmatrix} \neq 0$$

evaluate at $t=0$

$y_i(t)$ → linearly independent

initial condition $t=0$: y_0

Lecture 11

1st order

$\alpha_0 \cdot y' + \alpha_1 \cdot y = g(t)$

$y' \equiv \frac{dy}{dt}$

$\alpha_0 \neq 0$

Assume $\alpha_1 = 0$

$y' = \frac{1}{\alpha_0} \cdot g(t) \Leftrightarrow$

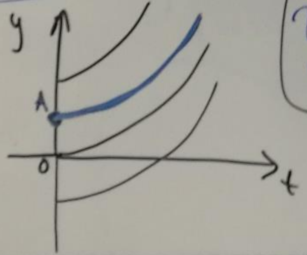
$\Leftrightarrow \int dy = \frac{1}{\alpha_0} \int g(t)\, dt \Leftrightarrow$

$\Leftrightarrow y(t) = \frac{1}{\alpha_0} \int g(t)\, dt + C$

if I know

$t=0 : y_0$

$y_0 = A \cdot e^{-\beta \cdot 0} = A$



Homogeneous

$\frac{\alpha_0}{\alpha_0} \cdot y' + \frac{\alpha_1}{\alpha_0} y = 0$

$\beta \equiv \frac{\alpha_1}{\alpha_0}$

$y' + \beta \cdot y = 0 \Leftrightarrow$

$\Leftrightarrow \boxed{y' = -\beta \cdot y}$

if $(-\beta) = 1$

$y' = y$

Wise guess

$y = e^{\lambda \cdot t}$

$y' = \lambda \cdot e^{\lambda \cdot t}$

$$\boxed{y = A \cdot e^{-\beta \cdot t}}$$

Integration

$y' + \beta \cdot y = 0 \Leftrightarrow$

$\frac{d \ln y}{dt} = \frac{1}{y} \frac{dy}{dt}$

$\Leftrightarrow \frac{y'}{y} + \beta = 0 \Leftrightarrow \frac{y'}{y} = -\beta \Leftrightarrow$

$\Leftrightarrow \ln y = -\beta t + C$

$\left\{ \frac{1}{y} \frac{dy}{dt} = -\beta \right\}$

$\lambda \cdot e^{\lambda \cdot t} + \beta \cdot e^{\lambda \cdot t} = 0 \Leftrightarrow$

$\Leftrightarrow e^{\lambda \cdot t} [\lambda + \beta] = 0 \Leftrightarrow$

$\Leftrightarrow \lambda + \beta = 0 \Leftrightarrow \lambda = -\beta$

If I know $t = t^*$, $y(t^*) = y^*$

$y^* = A \cdot e^{-\beta \cdot t^*} \Leftrightarrow$

$\Leftrightarrow A = \frac{y^*}{e^{-\beta \cdot t^*}}$

Lecture 11

1st order

$\alpha_0 \cdot y' + \alpha_1 \cdot y = g(t)$

$y' \equiv \frac{dy}{dt}$

$\alpha_0 \neq 0$

Homogeneous
& solution

$y = A \cdot e^{(-B) \cdot t}$

Dynamics $\rightarrow -B$

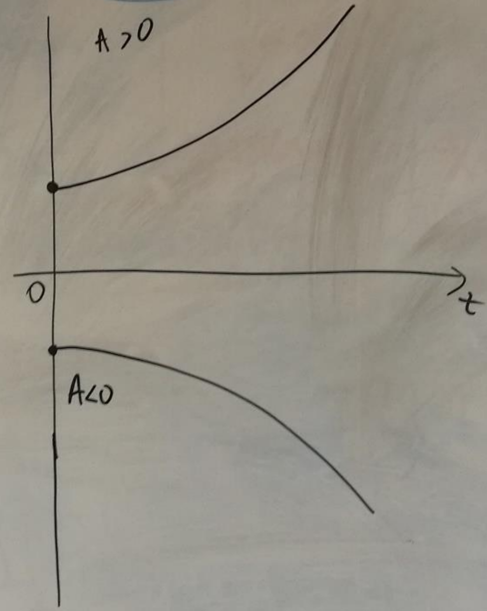
For stability
$-B < 0$

$e^{\lambda \cdot t} \cdot [\lambda + B] = 0$

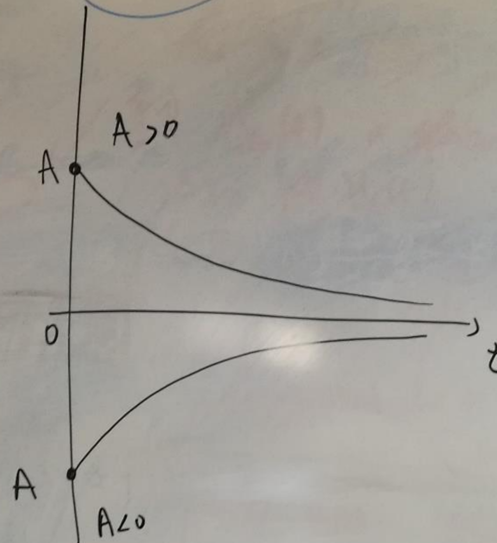
characteristic equation

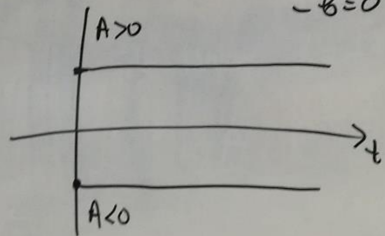
3

$-B > 0$ Explosive



$-B < 0$ Stability





Lecture 11

4

1st order

$\alpha_0 \cdot y' + \alpha_1 \cdot y = g(t) \rightarrow$ Non Homogeneous [Particular Solution]

$y' \equiv \frac{dy}{dt}$

$\alpha_0 \neq 0$

Homogeneous solution

$y = A \cdot e^{(-\beta) \cdot t}$

Dynamics $\rightarrow -\beta$

For stability $-\beta < 0$

$e^{\lambda \cdot t} \cdot [\lambda + \beta] = 0$

characteristic equation

Non Homogeneous [Particular Solution]

Guess $\bar{y}$ of the same algebraic form as $g(t)$

$\rightarrow g(t) = B \cdot e^{m \cdot t}$

Guess $\bar{y} = \Gamma \cdot e^{m \cdot t}$

Method of undetermined coefficients

$\bar{y}' = m \cdot \Gamma \cdot e^{m \cdot t}$

$\alpha_0 \cdot m \cdot \Gamma e^{m \cdot t} + \alpha_1 \cdot \Gamma \cdot e^{m \cdot t} = B \cdot e^{m \cdot t}$

$\Leftrightarrow \Gamma \cdot (\alpha_0 \cdot m + \alpha_1) \cdot \cancel{e^{m \cdot t}} = B \cdot \cancel{e^{m \cdot t}} \Leftrightarrow$

$\Leftrightarrow \Gamma = \frac{B}{\alpha_0 \cdot m + \alpha_1}$ iff $\alpha_0 \cdot m + \alpha_1 \neq 0$

If $\alpha_0 \cdot m + 1 = 0 \rightarrow$ Then Guess $\bar{y} = t \cdot \Gamma \cdot e^{m \cdot t} \rightarrow$ Find Γ

$g(t) \rightarrow$ a generic function of time

Lagrange method of variation

I know the solution to the homogeneous: $y = A \cdot e^{-\beta \cdot t}$

Assume for the solution to the non-homogeneous: $y(t) = A(t) \cdot e^{-\beta \cdot t}$ (I)

undetermined function of time

$y' = A'(t) \cdot e^{-\beta \cdot t} + (-\beta) \cdot A(t) \cdot e^{-\beta t} \Leftrightarrow$

$\Leftrightarrow y' = A'(t) \cdot e^{-\beta t} - \beta \cdot A(t) \cdot e^{-\beta t}$ (II)

$\frac{\alpha_0 \cdot y'}{\alpha_0} + \frac{\alpha_1 \cdot y}{\alpha_0} = g(t) \Leftrightarrow$

$y' + \beta y = \frac{1}{\alpha_0} \cdot g(t) \overset{(I)(II)}{\Leftrightarrow} A'(t) \cdot e^{-\beta t} - \cancel{\beta \cdot A(t) \cdot e^{-\beta t}} + \cancel{\beta \cdot A(t) \cdot e^{-\beta t}} = \frac{1}{\alpha_0} \cdot g(t) \Leftrightarrow$

$\Leftrightarrow A'(t) \cdot e^{-\beta t} = \frac{1}{\alpha_0} g(t) \Leftrightarrow \frac{dA(t)}{dt} = \frac{1}{\alpha_0} g(t) \Leftrightarrow$

$\Leftrightarrow$ Integrate $\Leftrightarrow$

$\Leftrightarrow A(t) = \frac{1}{\alpha_0} \int g(t) \cdot e^{\beta t} dt + \Gamma \Leftrightarrow$ substitute in (I)

$y(t) = \underbrace{\boxed{\Gamma \cdot e^{-\beta \cdot t}}}_{\text{General solution of the homogeneous}} + \underbrace{\boxed{\frac{1}{\alpha_0} e^{-\beta \cdot t} \int g(t) e^{\beta t} dt}}_{\text{Particular solution}}$

$\bar{y}$

Lecture 11

2n order

$\alpha_0 \cdot y'' + \alpha_1 \cdot y' + \alpha_2 \cdot y = g(t)$

$\alpha_0 \neq 0$

Higher order n

$\alpha_0 \cdot y^{(n)} + \dots + \alpha_{n-1} \cdot y' + \alpha_n y = g(t)$

Homogeneous

$\frac{\alpha_0 \cdot y''}{\alpha_0} + \underbrace{\frac{\alpha_1}{\alpha_0}}_{b_1} \cdot y' + \underbrace{\frac{\alpha_2}{\alpha}}_{b_2} \cdot y = 0$

$y'' + b_1 \cdot y' + b_2 \cdot y = 0$

Wise guess

$$\left.\begin{aligned} y &= e^{\lambda t} \\ y' &= \lambda \cdot e^{\lambda t} \\ y'' &= \lambda^2 \cdot e^{\lambda t} \end{aligned}\right\} \quad \lambda^2 \cdot e^{\lambda t} + b_1 \cdot \lambda \cdot e^{\lambda t} + b_2 \cdot e^{\lambda t} = 0 \Leftrightarrow e^{\lambda t} \cdot [\lambda^2 + b_1 \cdot \lambda + b_2] = 0$$

$\lambda^2 + b_1 \cdot \lambda + b_2 = 0 \rightarrow \lambda_1, \lambda_2$

↓

Discriminant Δ

$\Delta > 0 \rightarrow \lambda_1, \lambda_2$ real distinct roots

$y(t) = A_1 \cdot e^{\lambda_1 t} + A_2 \cdot e^{\lambda_2 \cdot t}$

$\Delta = 0 \rightarrow \lambda_1 = \lambda_2 = \lambda^* \in \mathbb{R}$

$y(t) = A_1 \cdot e^{\lambda^* \cdot t} + A_2 \cdot t \, e^{\lambda^* \cdot t}$

$\Delta < 0 \rightarrow$ two complex conjugate roots

Stability

↓

Depends on the sign of the dominant root ($\Leftrightarrow$)

$\Delta > 0$ Dominant root $= \max\{\lambda_1, \lambda_2\} < 0$

Determination of A_1, A_2

If I know y at t^* : $y(t = t^*) = y^*$

$y'(t = t^*) = y'^*$

Evaluate at $t = t^*$

$y^* = A_1 \cdot e^{\lambda_1 \cdot t^*} + A_2 \cdot e^{\lambda_2 \cdot t^*}$

$y^{*\prime} = A_1 \cdot \lambda_1 \cdot e^{\lambda_1 t^*} + A_2 \cdot \lambda_2 \cdot e^{\lambda_2 \cdot t^*}$

Calculate A_1, A_2