Lecture 10

1

$$y_{t+1} = \alpha_{11} \cdot y_t + \alpha_{12} \cdot z_t$$

$$z_{t+1} = \alpha_{21} \cdot y_t + \alpha_{22} \cdot z_t$$

Solution

$$y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$$

$$z_t = A_1 \alpha_1 \cdot \lambda_1^t + A_2 \cdot \alpha_2 \cdot \lambda_2^t$$

where A_1, A_2 arbitrary constants (General)

$$\alpha_2 = \frac{\lambda_1 - \alpha_{11}}{\alpha_{12}}$$

$$\alpha_2 = \frac{\lambda_2 - \alpha_{11}}{\alpha_{12}}$$

Stability Criterion $|\lambda_1|, |\lambda_2| \in (0,1)$

$\max\{|\lambda_1|, |\lambda_2|\} \in (0,1)$

It all depends on the nature of the variables [Theory]

λ_1: stable root

λ_2: unstable $(|\lambda_2| > 1)$

Explosive unless $A_2 = 0$

Variables

I cannot choose...

predetermined (state)

or

control (jump)

I can choose the initial condition so as to eliminate the effect of the unstable root!

{saddle path stability}

Lecture 10

2

$n\times n$ System of linear 1st order ODE

$$y_{t+1} = A\cdot y_t + G(t)$$

$$y_t \equiv \begin{bmatrix} y_{1_t} \\ y_{2_t} \\ \vdots \\ y_{n_t} \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}_{n\times n}$$

$$G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

Homogeneous

$$\underset{n\times 1}{y_{t+1}} = \underset{n\times n}{A}\cdot \underset{n\times 1}{y_t} \Leftarrow a\cdot\lambda^{t+1} = A\cdot a\lambda^t \Leftarrow a\cdot\lambda\cdot\lambda^t = A\cdot a\cdot\lambda^t \Leftarrow$$

$$\Leftarrow A\cdot a\cdot\lambda^t - a\cdot\lambda\cdot\lambda^t = \mathbb{0}_{n\times 1} \Leftarrow$$

$$\lambda^t\cdot[A-\lambda\cdot I]\cdot a = \mathbb{0} \Leftarrow$$

$$\Leftarrow [\underset{n\times n}{A} - \lambda\cdot \underset{n\times n}{I}]\cdot \underset{n\times 1}{a} = \mathbb{0}_{n\times 1}$$

Guess:

$$\boxed{\underset{n\times 1}{y_t} = \underset{n\times 1}{a}\cdot\lambda^t} = \begin{bmatrix} a_1\cdot\lambda^t \\ a_2\cdot\lambda^t \\ \vdots \\ a_n\cdot\lambda^t \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}_{n\times 1}$$

Homogeneous Linear System

For this system to have a non-trivial solution ($a \neq \mathbb{0}$)

$$\det(A-\lambda\cdot I) = 0$$

$$|A-\lambda\cdot I| = \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\lambda & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}-\lambda \end{vmatrix} = 0 \Leftarrow f(\lambda) = 0$$

n'th order polynomial

$$c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n = 0$$

Characteristic Polynomial of matrix A

n-Roots → Eigenvalues of A
(Characteristic values)

Lecture 10

3

$n \times n$ System of linear 1st order ODE

$$y_{t+1} = A \cdot y_t + G(t)$$

$$y_t \equiv \begin{bmatrix} y_{1_t} \\ y_{2_t} \\ \vdots \\ y_{n_t} \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}_{n \times n}$$

$$G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

$$\boxed{y_t = a \cdot \lambda^t} = \begin{bmatrix} a_1 \cdot \lambda^t \\ a_2 \cdot \lambda^t \\ \vdots \\ a_n \cdot \lambda^t \end{bmatrix} \qquad a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$[\underset{n\times n}{A} - \lambda \cdot \underset{n\times n}{I}] \cdot \underset{n \times 1}{a} = \underset{n \times 1}{0}$$

$$|A - \lambda I| = 0$$

↳ roots: eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_n$

Stability Criterion

Dominant eigenvalue $\max_i \{|\lambda_i|\} < 1$

Solution [n distinct real roots]

$$y_{1_t} = A_1 \cdot a_1^{(1)} \cdot \lambda_1^t + A_2 \cdot a_1^{(2)} \cdot \lambda_2^t + \dots + A_n \cdot a_1^{(n)} \cdot \lambda_n^t$$

$$y_{2_t} = A_1 \cdot a_2^{(1)} \cdot \lambda_1^t + A_2 \cdot a_2^{(2)} \cdot \lambda_2^t + \dots + A_n \cdot a_2^{(n)} \cdot \lambda_n^t$$

$$\vdots$$

$$y_{n_t} = A_1 \cdot a_n^{(1)} \cdot \lambda_1^t + A_2 \cdot a_n^{(2)} \cdot \lambda_2^t + \dots + A_n \cdot a_n^{(n)} \cdot \lambda_n^t$$

Determination of $A_1, A_2, \dots, A_n$

Information: "Initial" Condition for $t=0$

$$t=0: \bar{y}_0 = \begin{bmatrix} \bar{y}_{1_0} \\ \bar{y}_{2_0} \\ \vdots \\ \bar{y}_{n_0} \end{bmatrix}$$

Lecture 10

$n \times n$ System of linear 1st order ODE

$$y_{t+1} = A \cdot y_t + G(t)$$

$$y_t \equiv \begin{bmatrix} y_{1_t} \\ y_{2_t} \\ \vdots \\ y_{n_t} \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}_{n \times n}$$

$$G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

4

$$\left[A_{n \times n} - \lambda \cdot I_{n \times n}\right] \cdot a_{n \times 1} = \mathbb{O}_{n \times 1}$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boxed{y_t = a \cdot \lambda^t} = \begin{bmatrix} a_1 \cdot \lambda^t \\ a_2 \cdot \lambda^t \\ \vdots \\ a_n \cdot \lambda^t \end{bmatrix}$$

Guess for the solution

$$|A - \lambda I| = 0$$

↳ roots : eigenvalues

$\lambda_1, \lambda_2, \dots, \lambda_n$ [n distinct real roots]

Stability Criterion

Dominant eigenvalue $\max_i \{|\lambda_i|\} < 1$

Determination of the arbitrary constants → I know $\bar{y}_0$

$$\bar{y}_{1_0} = A_1 \cdot a_1^{(1)} \cdot \lambda_1^{0} + A_2 \cdot a_1^{(2)} \cdot \lambda_2^{0} + \dots + A_n \cdot a_1^{(n)} \cdot \lambda_n^{0}$$

$$\bar{y}_{2_0} = A_1 \cdot a_2^{(1)} \cdot \lambda_1^{0} + A_2 \cdot a_2^{(2)} \cdot \lambda_2^{0} + \dots + A_n \cdot a_2^{(n)} \cdot \lambda_n^{0}$$

$$\vdots$$

$$\bar{y}_{n_0} = A_1 \cdot a_n^{(1)} \cdot \lambda_1^{0} + A_2 \cdot a_n^{(2)} \cdot \lambda_2^{0} + \dots + A_n \cdot a_n^{(n)} \cdot \lambda_n^{0}$$

$\lambda_1^0 = 1$, $\lambda_2^0 = 1$, $\dots$, $\lambda_n^0 = 1$

$(\Leftrightarrow)$

Determination of $A_1, A_2, \dots, A_n$

Information : "Initial" Condition for $t = 0$

$$t = 0 : \bar{y}_0 = \begin{bmatrix} \bar{y}_{1_0} \\ \bar{y}_{2_0} \\ \vdots \\ \bar{y}_{n_0} \end{bmatrix}$$

$$(\Leftrightarrow) \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & \cdots & a_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ a_n^{(1)} & a_n^{(2)} & \cdots & a_n^{(n)} \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} \bar{y}_{1_0} \\ \bar{y}_{2_0} \\ \vdots \\ \bar{y}_{n_0} \end{bmatrix} (\Leftarrow)$$

$$\alpha \equiv \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix}$$

$$(\Leftrightarrow) \; \bar{y}_{0_{n \times 1}} = V_{n \times n} \cdot \alpha_{n \times 1} \; (\Leftarrow)$$

$$(\Leftarrow) \quad \boxed{\alpha = V^{-1} \cdot \bar{y}_0}$$

Lecture 10

6

$n \times n$ System of linear 1st order ODE

$$y_{t+1} = A \cdot y_t + G(t)$$

$$y_t \equiv \begin{bmatrix} y_{1_t} \\ y_{2_t} \\ \vdots \\ y_{n_t} \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}_{n \times n}$$

$$G(t) = \begin{bmatrix} g_1(t) \\ g_2(t) \\ \vdots \\ g_n(t) \end{bmatrix}$$

$$\left[A_{n\times n} - \lambda \cdot I_{n \times n}\right] \cdot a_{n \times 1} = \emptyset_{n \times 1}$$

$$|A - \lambda I| = 0$$

$$a = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boxed{y_t = a \cdot \lambda^t} = \begin{bmatrix} a_1 \cdot \lambda^t \\ a_2 \cdot \lambda^t \\ \vdots \\ a_n \cdot \lambda^t \end{bmatrix}$$

Guess for the solution

$$V = \begin{bmatrix} a_1^{(1)} & a_1^{(2)} & \cdots & a_1^{(n)} \\ a_2^{(1)} & a_2^{(2)} & \cdots & a_2^{(n)} \\ \vdots & \vdots & & \vdots \\ a_n^{(1)} & a_n^{(2)} & \cdots & a_n^{(n)} \end{bmatrix}$$

eigenvector (i) corresponding to λ_i $a^{(i)}$: $\left[A - \lambda_i I\right] a^{(i)} = \emptyset$ $\forall i$

Matrix of eigenvectors of matrix A

$\lambda_1, \lambda_2, \dots, \lambda_n \rightarrow n$ distinct real roots

$\Rightarrow$ To distinct roots correspond linearly independent eigenvectors

$\hookrightarrow \exists V^{-1}$

Lecture 10

Solution

$$y_t = A^t \cdot c$$

Diagonalization

$$A = V \cdot \Lambda \cdot V^{-1}$$

$$A^t = V \cdot \Lambda^t \cdot V^{-1}$$

$$y_t = V \cdot \Lambda^t \cdot V^{-1} \cdot c \Leftarrow$$

$$\Longleftrightarrow y_t = V \cdot \Lambda^t \cdot V^{-1} c$$

arbitrary constant, let α

$$\Leftarrow \boxed{y_t = V \cdot \Lambda^t \cdot \alpha}$$

Stationarity $\rightarrow \lambda_i$

If $|\lambda_i| < 1$

$$\lim_{t \to 0} \Lambda^t = \mathbb{0}_{n \times n} \qquad \lim_{t \to 0} y_t = \mathbb{0}_{n \times 1}$$

Directly $\rightarrow$ matrix solution

Homogeneous System

$$\underset{n\times 1}{y_{t+1}} = \underset{n\times n}{A} \cdot \underset{n\times 1}{y_t}$$

$y_t = A^t \cdot c$, $y_{t+1} = A^{t+1} \cdot c$ $\quad A^{t+1} \cdot c = A \cdot A^t \cdot c$ ✓ Satisfied!

We have found scalar

$$y_{t+1} = \alpha \cdot y_t$$

Solution $y_t = c \cdot \alpha^t$ (arbitrary constant)

Apply this to the case above of a system

$$y_1 = A \cdot c$$

$$y_2 = A^2 \cdot c$$

$$\vdots$$

$$y_s = A^s \cdot c$$

As before $\rightarrow$ $\lambda_1, \lambda_2, \dots, \lambda_n$ distinct real roots

Diagonalization of matrix A :

$$\Lambda = V^{-1} A \cdot V \Leftarrow,$$

$$\Leftarrow, A = V \cdot \Lambda \cdot V^{-1} \Leftarrow,$$

$$\Leftarrow, A^2 = (V \cdot \Lambda \cdot V^{-1})(V \Lambda V^{-1}) = V \cdot \Lambda^2 \cdot V^{-1}$$

$$\Leftarrow, A^3 = (V \cdot \Lambda \cdot V^{-1})(V \Lambda^2 \cdot V^{-1}) = V \cdot \Lambda^3 \cdot V^{-1}$$

: Solution $\rightarrow$ $\underset{n\times 1}{y_t} = \underset{n\times n}{A^t} \cdot c$

$$A^t = V \cdot \Lambda^t \cdot V^{-1}$$

value of n arbitrary constants

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$\Lambda^n = \mathrm{diag}(\lambda_1^n, \lambda_2^n, \dots, \lambda_n^n)$$

(Similarity transformation)

Λ and A are called "similar" matrices

There exists matrix V (matrix of eigenvectors)

$$\Lambda = V^{-1} \cdot A \cdot V$$

$$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & \emptyset \\ & \lambda_2 & \\ \emptyset & & \ddots \ \lambda_n \end{bmatrix} \qquad \Lambda^t = \begin{bmatrix} \lambda_1^t & & \emptyset \\ & \lambda_2^t & \\ \emptyset & & \ddots \ \lambda_n^t \end{bmatrix}$$

7