Lecture 9

1

n-th order ODE

$$c_0 \cdot y_t + c_1 \cdot y_{t-1} + \dots + c_n \cdot y_{t-n} = g(t)$$

$c_0 \neq 0$

$c_n \neq 0$

Homogeneous

$$c_0 \cdot y_t + c_1 \cdot y_{t-1} + \dots + c_n \cdot y_{t-n} = 0$$

Wise Guess

$$y_t = \lambda^t$$

$$c_0 \cdot \lambda^t + c_1 \cdot \lambda^{t-1} + \dots + c_n \cdot \underline{\underline{\lambda}}^{t-n} = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda^{t-n} \cdot \underbrace{\left[c_0 \cdot \lambda^n + c_1 \cdot \lambda^{n-1} + \dots + c_n \right]}_{\text{Characteristic function}} = 0$$

$$\frac{c_0}{c_0} \cdot \lambda^n + \frac{c_1}{c_0} \lambda^{n-1} + \dots + \frac{c_n}{c_0} = 0 \Leftrightarrow$$

$$\Leftrightarrow \lambda^n + \alpha_1 \cdot \lambda^{n-1} + \dots + \alpha_0 = 0$$

n roots

- real distinct
- multiplicity
- conjugate complex

Let n roots

$\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$

$\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j \ \forall i \neq j$

$$y_t = A_1 \cdot \lambda_1^t + \dots + A_n \cdot \lambda_n^t$$

Dominant root $=$

$= \max\{|\lambda_i|, i=1,\dots,n\}$

If < 1 stabilizing

If > 1 explosive

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n-th order ODE

$$C_0 \cdot y_t + C_1 \cdot y_{t-1} + \dots + C_n \cdot y_{t-n} = g(t)$$

$C_0 \neq 0$

$C_n \neq 0$

Particular Solution

$$g(t) = X_t$$

$$x_t \equiv \frac{X_t}{C_0}$$

$$y_t + \alpha_1 \cdot y_{t-1} + \dots + \alpha_n \cdot y_{t-n} = x_t$$

$$\alpha_i = \frac{C_i}{C_0}, \quad i = 1, \dots, n$$

In terms of operators

$$y_t + \alpha_1 L y_t + \alpha_2 L^2 y_t + \dots + \alpha_n L^n y_t = x_t \Leftrightarrow$$

$$1 + \alpha_1 L + \dots + \alpha_n L^n =$$

$$= L^n \left[L^{-n} + \alpha_1 L^{-(n-1)} + \dots + \alpha_n \right] =$$

$$= L^n \left[F^n + \alpha_1 F^{n-1} + \dots + \alpha_n \right] \Leftrightarrow$$

$$L^{-n} \equiv F^n \equiv x_{t+n}$$

$x_{t-(-n)}$

If φ_i roots $[\ ] = 0$

These are the roots of the characteristic function of the homogeneous $\varphi_i = \lambda_i$

ASSUMPTION: n distinct real roots

$$\Leftrightarrow F^n + \alpha_1 F^{n-1} + \dots + \alpha_n = (F - \varphi_1)(F - \varphi_2) \cdots (F - \varphi_n)$$

Therefore

$$L^n \cdot \left[F^n + \alpha_1 F^{n-1} + \dots + \alpha_n \right] =$$

$$= L^n (F - \lambda_1)(F - \lambda_2) \cdots (F - \lambda_n) \Leftrightarrow$$

$$LF = LL^{-1} = 1$$

$$\Leftrightarrow \left[1 + \alpha_1 L + \alpha_2 L^2 + \dots + \alpha_n L^n \right] y_t = x_t$$

$$\Leftrightarrow y_t = \frac{x_t}{\left[1 + \alpha_1 L + \dots + \alpha_n L^n \right]}$$

$$\Leftrightarrow 1 + \alpha_1 L + \dots + \alpha_n L^n =$$

$$= (1 - \lambda_1 L)(1 - \lambda_2 L) \cdots (1 - \lambda_n L) \Leftrightarrow$$

$$\frac{1}{[1 + \alpha_1 L + \dots \alpha_n L^n]} = \frac{1}{(1 - \lambda_1 L) \cdots (1 - \lambda_n L)} =$$

$$= \frac{\theta_1}{1 - \lambda_1 L} + \dots + \frac{\theta_n}{1 - \lambda_n L} \Leftrightarrow$$

$$\Leftrightarrow y_t = \sum_{i=1}^{n} \frac{\theta_i}{1 - \lambda_i L} x_t$$

i-th term: $\frac{\theta_i}{1 - \lambda_i L} x_t$

$|\lambda_i| < 1$ Backward solution

$|\lambda_i| > 1$ Forward solution

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n-th order ODE

$$C_0 \cdot y_t + C_1 \cdot y_{t-1} + \dots + C_n \cdot y_{t-n} = g(t)$$

$C_0 \neq 0$
$C_n \neq 0$

Solution [n distinct real roots]

particular solution

$$y(t) = A_1 \cdot \lambda_1^t + \dots + A_n \cdot \lambda_n^t + \bar{y}_t$$

Pin down A_i, $i = 1, \dots, n$

I need information for the value of y_t at n points in time

e.g. $t = 0, 1, \dots, n-1$

Casorati

$$\left.\begin{aligned} t=0 &: y_0 = A_1 + \dots + A_n + \bar{y}_0 \\ t=1 &: y_1 = A_1 \cdot \lambda_1 + \dots + A_n \cdot \lambda_n + \bar{y}_1 \\ &\vdots \\ t=n-1 &: y_{n-1} = A_1 \cdot \lambda_1^{n-1} + \dots + A_n \cdot \lambda_n^{n-1} + \bar{y}_{n-1} \end{aligned}\right\} \begin{bmatrix} 1 & \dots & 1 \\ \lambda_1 & \dots & \lambda_n \\ \vdots & \ddots & \vdots \\ \lambda_1^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} A_1 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} y_0 - \bar{y}_0 \\ \vdots \\ y_{n-1} - \bar{y}_{n-1} \end{bmatrix}$$

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4

Systems of difference equations.

2×2 system of 1st order ODE in normal form

y, z

$$y_{t+1} = \alpha_{11} \cdot y_t + \alpha_{12} z_t + g_1(t)$$

$$z_{t+1} = \alpha_{21} \cdot y_t + \alpha_{22} \cdot z_t + g_2(t)$$

α_{12} and/or $\alpha_{21} \neq 0$

Homogeneous

The "cumbersome" way by substitution...

① $y_{t+1} = \alpha_{11} \cdot y_t + \alpha_{12} z_t$

② $z_{t+1} = \alpha_{21} \cdot y_t + \alpha_{22} \cdot z_t$

In matrix form

$$\begin{bmatrix} y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \cdot \begin{bmatrix} y_t \\ z_t \end{bmatrix}$$

① $\xrightarrow{\alpha_{12} \neq 0}$ $z_t = \frac{1}{\alpha_{12}} y_{t+1} - \frac{\alpha_{11}}{\alpha_{12}} \cdot y_t \Leftrightarrow z_{t+1} = \frac{1}{\alpha_{12}} y_{t+2} - \frac{\alpha_{11}}{\alpha_{12}} y_{t+1}$

Substitute z_t, z_{t+1} in ②:

$$\underbrace{\frac{1}{\alpha_{12}} y_{t+2} - \frac{\alpha_{11}}{\alpha_{12}} y_{t+1}}_{z_{t+1}} = \alpha_{21} \cdot y_t + \alpha_{22} \cdot \Big[\underbrace{\frac{1}{\alpha_{12}} y_{t+1} - \frac{\alpha_{11}}{\alpha_{12}} y_t}_{z_t}\Big] \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\alpha_{12}} y_{t+2} - \frac{\alpha_{11}}{\alpha_{12}} y_{t+1} - \frac{\alpha_{12}}{\alpha_{12}} \alpha_{21} \cdot y_t - \frac{\alpha_{22}}{\alpha_{12}} y_{t+1} + \alpha_{22} \cdot \frac{\alpha_{11}}{\alpha_{12}} y_t = 0$$

$$\Leftrightarrow y_{t+2} - (\alpha_{11} + \alpha_{22}) \cdot y_{t+1} - (\alpha_{12} \alpha_{21} - \alpha_{11} \cdot \alpha_{22}) y_t = 0$$

↑ 2nd order wrt y only

Characteristic function: $\lambda^2 - (\alpha_{11} + \alpha_{22}) \cdot \lambda - (\alpha_{12}\alpha_{21} - \alpha_{11}\alpha_{22}) = 0$

λ_1 λ_2

If $\lambda_1 = \lambda_2 = \lambda^*$

$y_t = (A_1 + A_2 \cdot t) \cdot \lambda^{*t}$

If $\lambda_i \in \mathbb{R}$

$\lambda_i \neq \lambda_j$ $i \neq j$

Solution $y_t = A_1 \lambda_1^t + A_2 \lambda_2^t$

Then, $z_t = \frac{1}{\alpha_{12}} y_{t+1} - \frac{\alpha_{11}}{\alpha_{12}} \cdot y_t = \frac{1}{\alpha_{12}} [A_1 \lambda_1^{t+1} + A_2 \lambda_2^{t+1}] - \frac{\alpha_{11}}{\alpha_{12}} \cdot [A_1 \lambda_1^t + A_2 \cdot \lambda_2^t] \Leftrightarrow$

$$\Leftrightarrow z_t = \left[\frac{1}{\alpha_{12}} \cdot A_1 \cdot \lambda_1 - \frac{\alpha_{11}}{\alpha_{12}} A_1\right] \cdot \lambda_1^t + \left[\frac{1}{\alpha_{12}} A_2 \cdot \lambda_2 - \frac{\alpha_{11}}{\alpha_{12}} A_2\right] \cdot \lambda_2^t$$

$$y_t = A_1 \cdot \lambda_1^t + A_2 \cdot \lambda_2^t$$

$$z_t = A_1 \cdot \left[\frac{\lambda_1 - \alpha_{11}}{\alpha_{12}}\right] \cdot \lambda_1^t + A_2 \cdot \left[\frac{\lambda_2 - \alpha_{11}}{\alpha_{12}}\right] \cdot \lambda_2^t$$

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$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} = \boxed{A}$$

Systems of difference equations.

2×2 system of 1st order ODE in normal form

y, z

Homogeneous

$$\left.\begin{aligned} y_{t+1} &= \alpha_{11}\cdot y_t + \alpha_{12} z_t \\ z_{t+1} &= \alpha_{21}\cdot y_t + \alpha_{22}\cdot z_t \end{aligned}\right\}$$

The "wise guess" approach

Wise Guess:

$$y(t) = a_1\cdot\lambda^t$$

$$z(t) = a_2\cdot\lambda^t$$

$$\left.\begin{aligned} a_1\cdot\lambda^{t+1} &= \alpha_{11}\cdot a_1\lambda^t + \alpha_{12}\cdot a_2\lambda^t \\ a_2\cdot\lambda^{t+1} &= \alpha_{21} a_1\lambda^t + \alpha_{22}\cdot a_2\cdot\lambda^t \end{aligned}\right\} \Leftrightarrow \begin{aligned} &\lambda^t\cdot[a_1\cdot\lambda - \alpha_{11} a_1 - \alpha_{12} a_2] = 0 \\ &\lambda^t\cdot[a_2\cdot\lambda - \alpha_{21}\cdot a_1 - \alpha_{22}\cdot a_2] = 0 \end{aligned}$$

$$\begin{bmatrix} y_{t+1} \\ z_{t+1} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} y_t \\ z_t \end{bmatrix} \Leftarrow \left.\begin{aligned} &\lambda^t\cdot[a_1\cdot(\lambda-\alpha_{11}) - \alpha_{12}\cdot a_2] = 0 \\ &\lambda^t\cdot[-\alpha_{21} a_1 + a_2\cdot(\lambda-\alpha_{22})] = 0 \end{aligned}\right\} \Leftrightarrow \begin{aligned} &\lambda^t\cdot[a_1\cdot(\alpha_{11}-\lambda) + \alpha_{12} a_2] = 0 \\ &\lambda^t\cdot[\alpha_{21}\cdot a_1 + a_2\cdot(\alpha_{22}-\lambda)] = 0 \end{aligned}$$

Then

$$\left.\begin{aligned} a_1\cdot(\alpha_{11}-\lambda) + \alpha_{12}\cdot a_2 &= 0 \\ \alpha_{21}\cdot a_1 + a_2\cdot(\alpha_{22}-\lambda) &= 0 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{21} & \alpha_{22}-\lambda \end{bmatrix}\cdot\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftarrow \begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{21} & \alpha_{22}-\lambda \end{vmatrix} = 0 \Leftrightarrow (\star)$$

Homogeneous linear system

If $A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$ $\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} \\ \alpha_{21} & \alpha_{22}-\lambda \end{vmatrix} = \det(A-\lambda\cdot I)$

$$(\star) \Leftrightarrow (\alpha_{11}-\lambda)(\alpha_{22}-\lambda) - \alpha_{12}\cdot\alpha_{21} = 0 \Leftrightarrow \dots$$

$$\Leftrightarrow \lambda^2 - (\alpha_{11}+\alpha_{22})\cdot\lambda + (\alpha_{11}\cdot\alpha_{22} - \alpha_{12}\alpha_{21}) = 0$$

characteristic polynomial of the 2nd order ODE wrt y we ended up in the previous approach

The roots are the same

$\det(A-\lambda I)=0$

This is the characteristic polynomial of A, and it's roots are the eigenvalues of A.

$$\begin{bmatrix} y_{t+1} \\ z_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}}_{A}\cdot\begin{bmatrix} y_t \\ z_t \end{bmatrix}$$

$x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$, $x_{t+1} = A\cdot x_t$