Lecture 7

Non Homogeneous Equation

$$C_n \cdot y_{t+n} + \dots + C_1 \cdot y_{t+1} + C_0 \cdot y_t = g(t)$$

Solution:

$$y(t) = f(t; A_1, A_2, \dots, A_n) + \bar{y}(t)$$

- $y(t)$ ↑ general solution of the non-homogeneous ODE
- $f(t; A_1, A_2, \dots, A_n)$ ↑ general solution of the homogeneous ODE (transition – deviations from the (long run) equilibrium)
- $\bar{y}(t)$ ↑ particular solution {Method of undetermined coefficients} ((long run) equilibrium)

For example

$$t = 0 \rightarrow y_0$$
$$t = 1 \rightarrow y_1$$
$$\vdots$$
$$t = n-1 \rightarrow y_{n-1}$$

Unown values (initial condition)

Solution n-th order homogeneous ODE ①

$y_1(t), y_2(t), \dots, y_n(t)$

$$\rightarrow y(t) = A_1 \cdot y_1(t) + \dots + A_n \cdot y_n(t)$$

$$y_0 = A_1 \cdot y_1(0) + \dots + A_n \cdot y_n(0)$$
$$y_1 = A_1 \cdot y_1(1) + \dots + A_n \cdot y_n(1)$$
$$\vdots$$
$$y_{n-1} = A_1 \cdot y_1(n-1) + \dots + A_n \cdot y_n(n-1)$$

↓

$$\begin{bmatrix} y_1(0) & \dots & y_n(0) \\ y_1(1) & \dots & y_n(1) \\ \vdots & & \\ y_1(n-1) & \dots & y_n(n-1) \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix}$$

↑ Linear System with a unique Solution

Lecture 7

1st Order ODE

$$C_1 \cdot y_t + C_0 \cdot y_{t-1} = g(t)$$

↗ known (or generic) function

$C_0 \neq 0$ and $C_1 \neq 0$

Homogeneous

Homogeneous ⟶ General solution of the homogeneous: $y(t) = f(t;A) = A \cdot (-B)^t$

$$C_1 \cdot y_t + C_0 \cdot y_{t-1} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{C_1}{C_1} \cdot y_t + \frac{C_0}{C_1} y_{t-1} = 0 \Leftrightarrow \qquad \left(\frac{C_0}{C_1} = B\right)$$

$$\Leftrightarrow y_t + B \cdot y_{t-1} = 0 \Leftrightarrow y_t = (-B) \cdot y_{t-1}$$

Wise guess: $y(t) = A \cdot (-B)^t$, $y(t-1) = A \cdot (-B)^{t-1}$

$$A \cdot (-B)^t + B \cdot A(-B)^{t-1} = A \cdot (-B)^t + A \cdot (-B)^{t-1+1} = 0 \checkmark$$

Determination of A:

2

$t = t^* \rightarrow y(t^*) = y^*$

$$y(t) = A \cdot (-B)^t \Leftrightarrow$$

$$\Leftrightarrow y(t^*) = y^* = A \cdot (-B)^{t^*} \Leftrightarrow A = \frac{y^*}{(-B)^{t^*}}$$

$t^* = 0$ (initial condition)

$y(0) = y^0 = A \cdot (-B)^0 = A$

Observe:

$$y_t = -B \cdot y_{t-1}$$

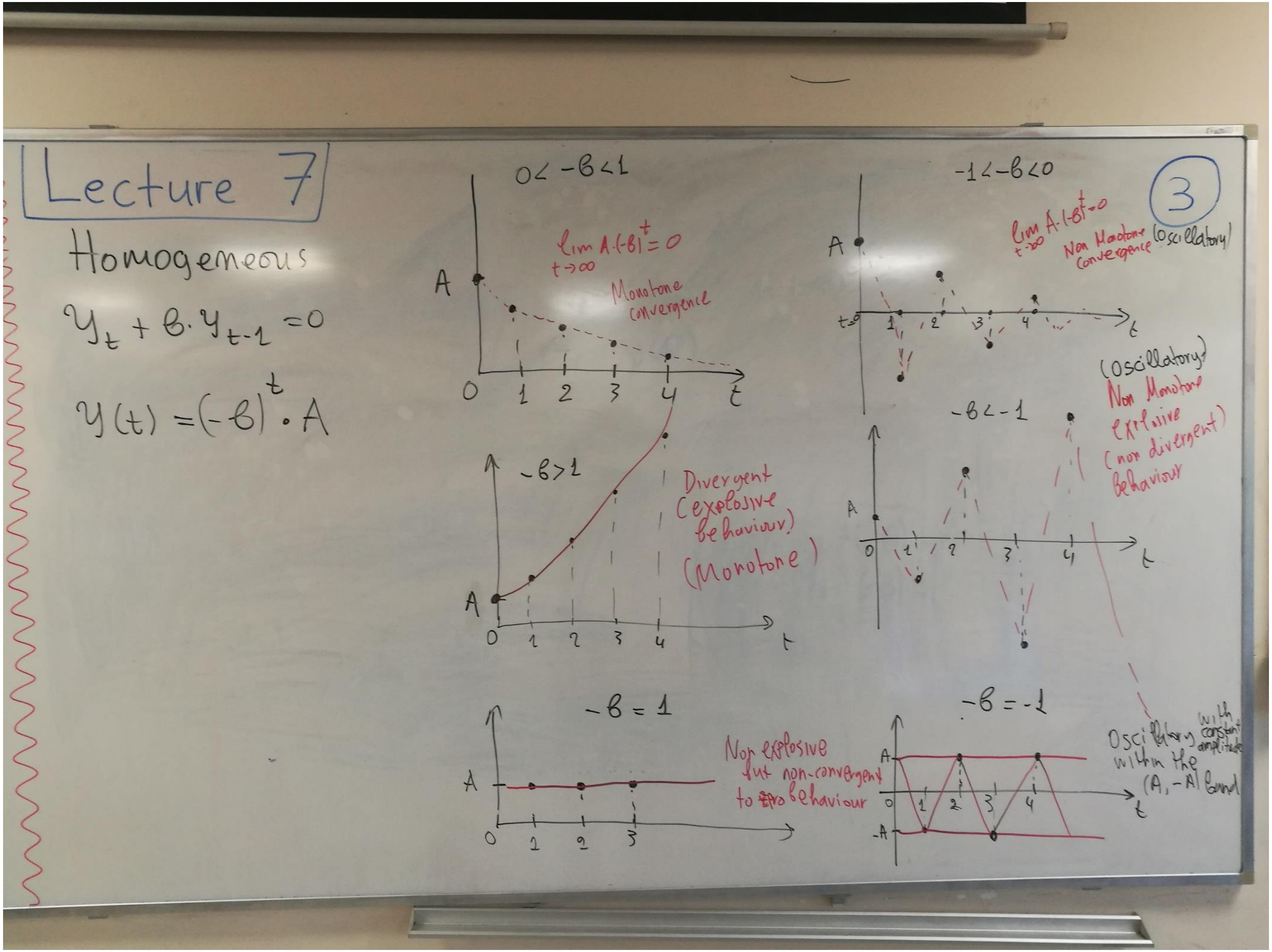
$t=1$: $y_1 = (-B) \cdot y_0$

$t=2$: $y_2 = (-B) \cdot y_1 = (-B) \cdot (-B) \cdot y_0 = (-B)^2 \cdot y_0$

$t=3$: $y_3 = (-B) \cdot y_2 = (-B) \cdot (-B)^2 \cdot y_0 = (-B)^3 \cdot y_0$

⋮

$t=S$: $y_S = (-B)^S \cdot y_0$


Lecture 7
3
Homogeneous
y_t + b·y_{t-1} = 0
y(t) = (-b)^t · A
0 < -b < 1
lim A·(-b)^t = 0
t→∞
Monotone convergence
-1 < -b < 0
lim A·(-b)^t = 0
t→∞
Non Monotone convergence (oscillatory)
-b > 1
Divergent (explosive behaviour)
(Monotone)
-b < -1
(Oscillatory)
Non Monotone
explosive
(non divergent)
Behaviour
-b = 1
Non explosive
but non-convergent
to zero behaviour
-b = -1
Oscillatory with constant amplitude
within the (A, -A) band

Lecture 7

3) Polynomial wrt t

$g(t) = \alpha_0 + \alpha_1 t + \alpha_2 t^2 \ldots$

4) Trigonometric function

5) Combination of (1), (2), (3), (4)

Non Homogeneous:

$C_1 \cdot y_t + C_0 \cdot y_{t-1} = g(t)$

Particular Solution $\bar{y}(t)$

6) $g(t) \rightarrow$ A generic function of time

I know the time series $g(0), g(1), \ldots$

I don't know the exact function $g(\cdot)$ that generates this time series

$K_{t+1} = (1-\delta) K_t + I_t \iff K_{t+1} - (1-\delta) K_t = I_t$ ← Time series of investment (exogenous variable)

$K_{t+1} - (1-\delta) K_t = 0$

$K(t) = A \cdot (1-\delta)^t$

1) $g(t) = \alpha$

$C_1 \cdot y_t + C_0 \cdot y_{t-1} = \alpha$

GUESS — Method of undetermined coefficients

$$\left.\begin{aligned} \bar{y}(t) &= \mu \\ \bar{y}(t-1) &= \mu \end{aligned}\right\} C_1 \cdot \mu + C_0 \cdot \mu = \alpha \iff$$

$\iff \mu \cdot (C_1 + C_0) = \alpha \iff$

$\iff \mu = \frac{\alpha}{C_1 + C_0} \rightarrow \bar{y} = \frac{\alpha}{C_1 + C_0}$, if $C_1 + C_0 \neq 0$

What happens when $C_1 + C_0 = 0$

Guess:

$$\left.\begin{aligned} \bar{y}(t) &= \mu \cdot t \\ \bar{y}(t-1) &= \mu \cdot (t-1) \end{aligned}\right\} C_1 \cdot \mu \cdot t + C_0 \cdot \mu \cdot (t-1) = \alpha \iff$$

$\iff (C_1 + C_0) \mu \cdot t - C_0 \cdot \mu = \alpha \iff \mu = -\frac{\alpha}{C_0}$

2) $g(t) = B \cdot \delta^t$

$C_1 \cdot y_t + C_0 \cdot y_{t-1} = B \cdot \delta^t$

Guess

$\bar{y}(t) = \Gamma \cdot \delta^t$

$\bar{y}(t-1) = \Gamma \cdot \delta^{t-1}$

$C_1 \cdot \Gamma \cdot \delta^t + C_0 \cdot \Gamma \cdot \delta^{t-1} = B \cdot \delta^t \iff \delta^t \cdot [C_1 \cdot \Gamma + C_0 \cdot \Gamma \cdot \delta^{-1}] = B \cdot \delta^t$

$\iff B = C_1 \cdot \Gamma + C_0 \cdot \Gamma \cdot \frac{1}{\delta} \iff B \cdot \delta = C_1 \cdot \Gamma \cdot \delta + C_0 \cdot \Gamma$

$\iff \Gamma = \frac{B \cdot \delta}{C_1 \cdot \delta + C_0}$, iff $C_1 \cdot \delta + C_0 \neq 0$

When $C_1 \cdot \delta + C_0 = 0$:

$\bar{y}(t) = \Gamma \cdot \delta^t \cdot t$

$\bar{y}(t-1) = \Gamma \cdot \delta^{t-1} \cdot (t-1)$

$\vdots$