Lecture 4

$$\max \int_0^\infty e^{-\theta \cdot t} \cdot g(x,u)\,dt \qquad x \to \text{state},\quad u \to \text{control}$$

$$\text{st} \quad \dot{x} = f(x,u)$$

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(Actual) Definition of the Hamiltonian: Present Value Hamiltonian, $\mu_t \to$ PV multiplier

$$\mathcal{H} = e^{-\theta \cdot t} \cdot g(x,u) + \mu_t \cdot f(x,u)$$

"Dot" Notation for time derivative

Current Value Hamiltonian

$$\dot{\mu}_t \equiv \frac{d\mu}{dt}, \quad \dot{\lambda}_t \equiv \frac{d\lambda}{dt}$$

Definition:

$$\overline{\mathcal{H}} = g(x,u) + \lambda_t \cdot f(x,u), \quad \lambda_t \to \text{CV multiplier}$$

Observe:

$$e^{\theta \cdot t} \cdot \mathcal{H} = e^{\theta \cdot t} \cdot e^{-\theta \cdot t} \cdot g(x,u) + \boxed{e^{\theta \cdot t} \cdot \mu_t} \cdot f(x,u) \Leftarrow,$$

$$\Leftarrow, \left\{ \lambda_t = e^{\theta \cdot t} \cdot \mu_t \text{ or } \mu_t = e^{-\theta t} \cdot \lambda_t \right\} \Leftarrow, e^{\theta \cdot t} \cdot \mathcal{H} = \overline{\mathcal{H}}$$

Observe also, $\mu_t = e^{-\theta \cdot t} \cdot \lambda_t \Leftarrow, \dot{\mu}_t = -\theta \cdot e^{-\theta \cdot t} \cdot \lambda_t + e^{-\theta \cdot t} \cdot \dot{\lambda}_t \Leftarrow, \dot{\mu}_t = e^{-\theta \cdot t} \cdot [-\theta \cdot \lambda_t + \dot{\lambda}_t]$

Lecture 4

PV : $\mathcal{H} = e^{-\theta \cdot t} \cdot g(x,u) + \mu \cdot f(x,u)$

CV : $\overline{\mathcal{H}} = g(x,u) + \lambda \cdot f(x,u)$

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PV multiplier : μ

CV " : λ

$$\overline{\mathcal{H}} = e^{\theta \cdot t} \cdot \mathcal{H}$$

or

$$\mathcal{H} = e^{-\theta \cdot t} \cdot \overline{\mathcal{H}}$$

$$\mu_t = e^{-\theta \cdot t} \cdot \lambda_t$$

$$\dot{\mu} = e^{-\theta \cdot t} \cdot [-\theta \cdot \lambda + \dot{\lambda}]$$

(Standard) FOC (PV-Hamiltonian)

1 $\frac{\partial \mathcal{H}}{\partial u} = 0 \Leftrightarrow e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial u} + \mu \cdot \frac{\partial f(x,u)}{\partial u} = 0$

2 $\frac{\partial \mathcal{H}}{\partial x} = -\dot{\mu} \Leftrightarrow e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial x} + \mu \cdot \frac{\partial f(x,u)}{\partial x} = -\dot{\mu}$

Observe:

$$\frac{\partial \overline{\mathcal{H}}}{\partial u} = \frac{\partial g(x,u)}{\partial u} + \lambda_t \cdot \frac{\partial f(x,u)}{\partial u}$$

From 1 : $e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial u} + \mu \cdot \frac{\partial f(x,u)}{\partial u} = 0$

$\Leftrightarrow \cancel{e^{-\theta \cdot t}} \cdot \frac{\partial g(x,u)}{\partial u} + \cancel{e^{-\theta t}} \cdot \lambda_t \cdot \frac{\partial f(x,u)}{\partial u} = 0$

$\Leftrightarrow \frac{\partial g(x,u)}{\partial u} + \lambda_t \cdot \frac{\partial f(x,u)}{\partial u} = 0 \Leftrightarrow \boxed{\frac{\partial \overline{\mathcal{H}}}{\partial u} = 0}$ 1'

Lecture 4

PV : $\mathcal{H} = e^{-\theta \cdot t} \cdot g(x,u) + \mu \cdot f(x,u)$

CV : $\overline{\mathcal{H}} = g(x,u) + \lambda \cdot f(x,u)$

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PV multiplier : μ

CV 〃 : λ

$$\overline{\mathcal{H}} = e^{\theta \cdot t} \cdot \mathcal{H}$$

or

$$\mathcal{H} = e^{-\theta \cdot t} \cdot \overline{\mathcal{H}}$$

$$\mu_t = e^{-\theta \cdot t} \cdot \lambda_t$$

$$\dot{\mu} = e^{-\theta \cdot t} \cdot [-\theta \cdot \lambda + \dot{\lambda}]$$

(Standard) FOC (PV-Hamiltonian)

① $\frac{\partial \mathcal{H}}{\partial u} = 0 \iff e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial u} + \mu \cdot \frac{\partial f(x,u)}{\partial u} = 0$

② $\frac{\partial \mathcal{H}}{\partial x} = -\dot{\mu} \iff e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial x} + \mu \cdot \frac{\partial f(x,u)}{\partial x} = -\dot{\mu}$

observe :

$$\frac{\partial \overline{\mathcal{H}}}{\partial x} = \frac{\partial g(x,u)}{\partial x} + \lambda \cdot \frac{\partial f(x,u)}{\partial x} =$$

$$= e^{\theta \cdot t} \cdot e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial x} + e^{\theta \cdot t} \cdot e^{-\theta \cdot t} \cdot \lambda \cdot \frac{\partial f(x,u)}{\partial x} =$$

$$= e^{\theta \cdot t} \cdot \left[e^{-\theta \cdot t} \cdot \frac{\partial g(x,u)}{\partial x} + \underbrace{e^{-\theta \cdot t} \cdot \lambda}_{\mu} \cdot \frac{\partial f(x,u)}{\partial x} \right] \iff$$

$$\iff \frac{\partial \overline{\mathcal{H}}}{\partial x} = e^{\theta \cdot t} \cdot \frac{\partial \mathcal{H}}{\partial x} \iff \frac{\partial \mathcal{H}}{\partial x} = e^{-\theta \cdot t} \cdot \frac{\partial \overline{\mathcal{H}}}{\partial x} \iff$$

$$\iff \{②\} \iff \frac{\partial \mathcal{H}}{\partial x} = \cancel{e^{-\theta \cdot t}} \cdot \frac{\partial \overline{\mathcal{H}}}{\partial x} = -\dot{\mu} = -\cancel{e^{-\theta \cdot t}} \cdot [-\theta \cdot \lambda + \dot{\lambda}]$$

CV-Hamiltonian

①' : $\frac{\partial \overline{\mathcal{H}}}{\partial u} = 0$

②' : $\frac{\partial \overline{\mathcal{H}}}{\partial x} = \theta \cdot \lambda - \dot{\lambda}$

Lecture 4

PV multiplier : μ
CV " : λ

$$\bar{\mathcal{H}} = e^{\theta \cdot t} \cdot \mathcal{H}$$

or

$$\mathcal{H} = e^{-\theta \cdot t} \cdot \bar{\mathcal{H}}$$

$$\mu_t = e^{-\theta \cdot t} \cdot \lambda_t$$

$$\dot{\mu} = e^{-\theta \cdot t} \cdot \left[-\theta \cdot \lambda + \dot{\lambda}\right]$$

PV - formulation
(standard Hamiltonian)

$$\mathcal{H} = e^{-\theta \cdot t} \cdot g(x,u) + \mu \cdot f(x,u)$$

FOC

1. $\dfrac{\partial \mathcal{H}}{\partial u} = 0$
2. $\dfrac{\partial \mathcal{H}}{\partial x} = -\dot{\mu}$

CV - formulation

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$$\bar{\mathcal{H}} = g(x,u) + \lambda \cdot f(x,u)$$

FOC

1'. $\dfrac{\partial \bar{\mathcal{H}}}{\partial u} = 0$
2'. $\dfrac{\partial \bar{\mathcal{H}}}{\partial x} = \theta \cdot \lambda - \dot{\lambda}$

EQUIVALENT

Lecture 4

PV-Hamiltonian VS CV-Hamiltonian in discrete time

$$\max \sum_{t=0}^{\infty} \beta^t g(x_t, u_t)$$

β: discount factor

observe in continuous time
the discount factor is $e^{-\theta \cdot t}$

$\theta \rightarrow$ rate of time preference
(interest rate)

$$\beta = \frac{1}{1+\theta}$$

s.t $x_{t+1} = f(x_t, u_t)$

PV

$$\mathcal{H} = \beta^t g(x_t, u_t) + \mu_{t+1} \cdot f(x_t, u_t)$$

FOC

$$\frac{\partial \mathcal{H}}{\partial u_t} = 0$$

$$\frac{\partial \mathcal{H}}{\partial x} = \mu_{t+1}$$

CV

$$\bar{\mathcal{H}} = g(x, u) + \lambda_{t+1} \cdot f(x_t, u_t)$$

FOC

$$\frac{\partial \bar{\mathcal{H}}}{\partial u_t} = 0$$

$$\frac{\partial \bar{\mathcal{H}}}{\partial x_t} = \beta^{-1} \cdot \lambda_{t+1}$$

Lecture 4

A Divergence back to Optimal control in continuous time :

Bang - Bang Solutions Case

[A e.g. case where the Hamiltonian is linear with respect to the control]

Problem → $\frac{\partial \mathcal{H}}{\partial u}$?

- Quickest way to fill a bath tub ?
- Quickest way of moving from point A to point B ?

$$\mathcal{H} = -1 + \lambda_1 x_2 + \lambda_2 u$$

Problem : Move the boat from point A to B at the shortest possible time

(5)

$$\boxed{\dot{x}_1 = x_2}$$

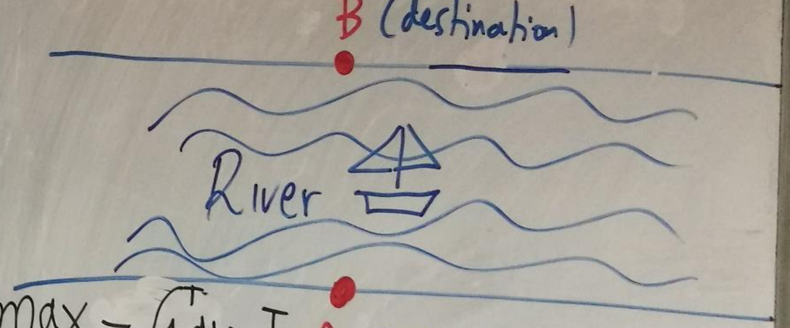
State $x1$: Distance left to reach B

State $x2$: The speed of the boat

Control u : Acceleration

$$\boxed{u = \dot{x}_2}$$

$u \in [-1, 1]$



$$\max -\int_0^T 1 \cdot dt = -T$$

$$\min_{u,T} \int_0^T 1 \cdot dt = T$$

s.t $\dot{x}_1 = x_2$ $\quad \lambda_1$

$\dot{x}_2 = u$ $\quad \lambda_2$

$u \in [-1, 1]$

Initial / Terminal Conditions — known

$x1(0)$ = Distance AB , $x2(0)$

$x1(T) = 0$, $x2(T) = 0$

Lecture 4

Dynamic Programming
(Discrete Time)

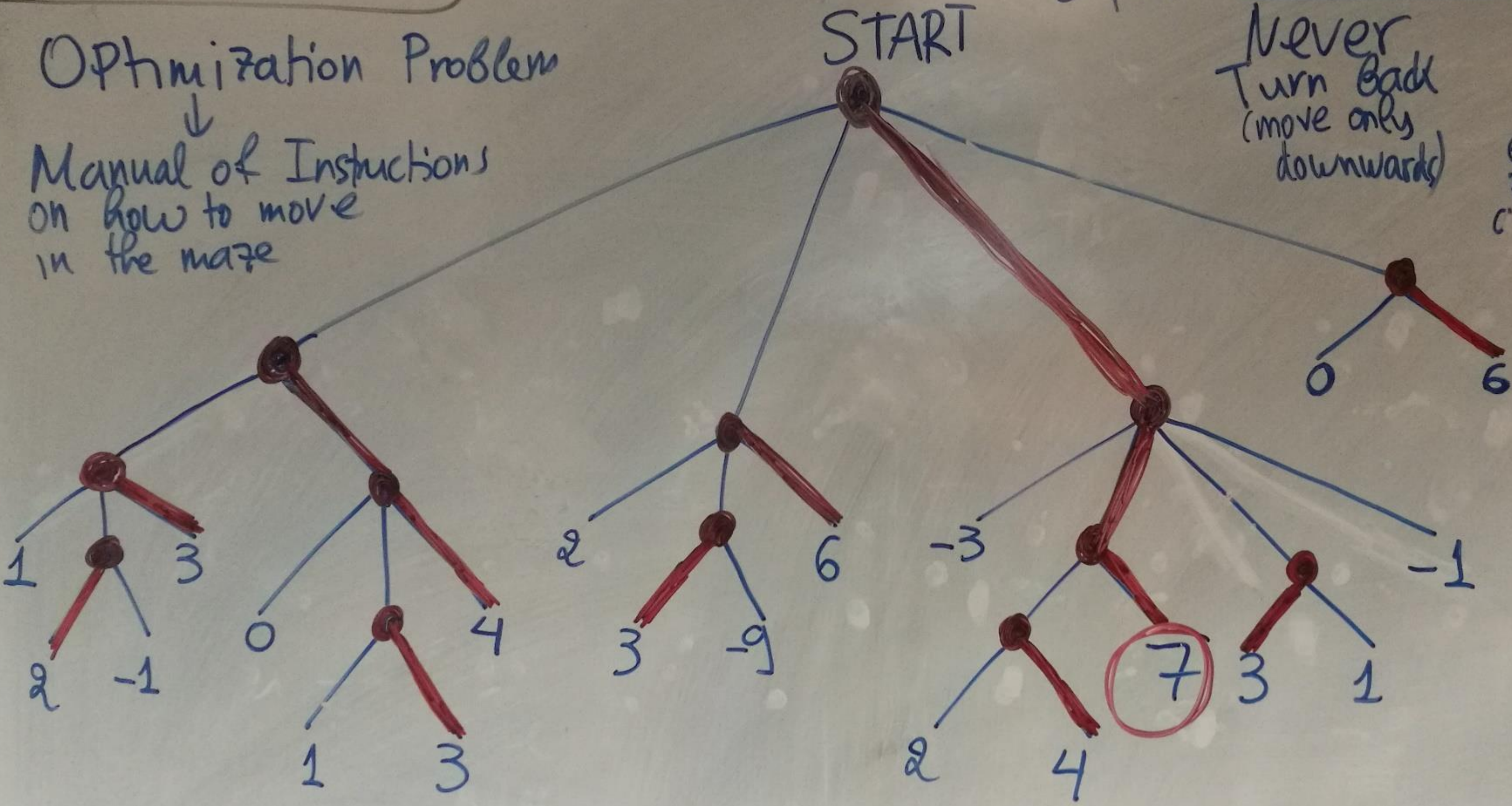
One entry point

Maze

7 many exit points
↓
each exit point implies a reward

Never Turn back
(move only downwards)

Optimization Problem
↓
Manual of Instructions on how to move in the maze



Recursiveness
↓
Backward induction
(Start at the bottom nods)