Lecture 1

(1)

Discrete time : $\varepsilon > 0$ (e.g. $\varepsilon = 1$) $\to t, t+\varepsilon$ [difference equation]

Continuous time : $\varepsilon \to 0$ [differential equations]

A. Discrete Time

$(+)$ $$\max_{\{u_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \cdot f(u_t, x_t)$$

$0 < \delta < 1$

s.t.

$\lambda_t \to$ $x_{t+1} - x_t = g(x_t, u_t)$, $t = 0, 1, \ldots$

multiplier

x_0 given

$x \to$ vector of STATE (predetermined) variables

$u \to$ " " CONTROL (state) "

+ FOC wrt μ_t $\left\{\frac{\partial \mathcal{L}}{\partial \mu_t} = 0\right\} \to$ Set of Constraints

A1. Lagrangean

$(*), (**), (+) \to$ system of difference equations w.r.t u, x, μ

$$\mathcal{L} = \sum_{t=0}^{\infty} \delta^t \cdot f(u_t, x_t) + \sum_{t=0}^{\infty} \lambda_t \cdot [g(x_t, u_t) + x_t - x_{t+1}]$$

$$= \sum_{t=0}^{\infty} \delta^t \cdot \left\{ f(u_t, x_t) + \mu_t \cdot [g(x_t, u_t) + x_t - x_{t+1}] \right\}$$

$\mu_t = \delta^{-t} \cdot \lambda_t$

F.O.C

u_t, $t \geq 0$: $\frac{\partial \mathcal{L}}{\partial u_t} = 0 \Leftarrow, \ \delta^t \cdot \left\{ \frac{\partial f(u_t, x_t)}{\partial u_t} + \mu_t \cdot \frac{\partial g(x_t, u_t)}{\partial u_t} \right\} = 0$

$\Leftarrow,$ $(*)$ $\boxed{\frac{\partial f(u_t, x_t)}{\partial u_t} + \mu_t \cdot \frac{\partial g(x_t, u_t)}{\partial u_t} = 0}$ $t \geq 0$

x_{t+1}, $t \geq 0$: $\frac{\partial \mathcal{L}}{\partial x_{t+1}} = 0 \Leftarrow, \ \delta^t \cdot \mu_t \cdot (-1) + \delta^{t+1} \cdot \left\{ \frac{\partial f(u_{t+1}, x_{t+1})}{\partial x_{t+1}} + \mu_{t+1} \cdot \left[\frac{\partial g(u_{t+1}, x_{t+1})}{\partial x_{t+1}} + 1 \right] \right\} = 0$

$(**)$ $\Leftarrow,$ $\boxed{\mu_t = \delta \cdot \left\{ \frac{\partial f(u_{t+1}, x_{t+1})}{\partial x_{t+1}} + \mu_{t+1} \cdot \left[1 + \frac{\partial g(u_{t+1}, x_{t+1})}{\partial x_{t+1}} \right] \right\}}$

Lecture 1

A. Discrete Time

$$\max_{\{u_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \cdot f(u_t, x_t) \qquad 0 < \delta < 1$$

s.t.

⊕ $x_{t+1} - x_t = g(x_t, u_t)$, $t = 0, 1, \ldots$

x_0 given

$x \rightarrow$ vector of STATE (predetermined) variables

$u \rightarrow$ " " CONTROL (state) "

[the ⊕ refers to the constraints]

A2. Dynamic Programming (2)

DEFINITION

Bellman Equation [functional equation]

$$V(x_t) = \max_{u_t, x_{t+1}} \left[f(u_t, x_t) + \delta \cdot V(x_{t+1}) \right] \quad (1)$$

s.t. $x_{t+1} - x_t = g(u_t, x_t) \Leftrightarrow$

$\Leftrightarrow x_{t+1} = x_t + g(u_t, x_t) \quad (2)$

(2) in (1):

$$V(x_t) = \max_{u_t} \left[f(u_t, x_t) + \delta \cdot V(\underbrace{x_t + g(u_t, x_t)}_{x_{t+1}}) \right] \quad (3)$$

F.O.C

u_t: [Derivative of (3) wrt u_t]

$$0 = \frac{\partial f(u_t, x_t)}{\partial u_t} + \delta \cdot \frac{\partial V(x_{t+1})}{\partial x_{t+1}} \cdot \frac{\partial x_{t+1}}{\partial u_t} \Leftrightarrow$$

$$\Leftrightarrow 0 = \frac{\partial f(u_t, x_t)}{\partial u_t} + \delta \cdot V'(x_{t+1}) \cdot \frac{\partial g(u_t, x_t)}{\partial u_t} \quad (4)$$

ENVELOPE CONDITION: [Derivative of (3) wrt x_t]

$$V'(x_t) = \frac{\partial f(u_t, x_t)}{\partial x_t} + \delta \cdot V'(x_{t+1}) \cdot \left[1 + \frac{\partial g(u_t, x_t)}{\partial x_t} \right] \quad (5)$$

Lecture 1) A2. Dynamic Programming (3)

Envelope Condition (▲)
+
Constraints (+)

} Dynamic System x, u

A. Discrete Time

$$\max_{\{u_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \cdot f(u_t, x_t)$$

$0 < \delta < 1$

s.t.

(+) $x_{t+1} - x_t = g(x_t, u_t)$, $t = 0, 1, \ldots$

x_0 given

$x \rightarrow$ vector of (predetermined) STATE variables

$u \rightarrow$ " " CONTROL (state) "

[the (+) refers to the constraints]

Use (4) to substitute away V' in (5)

(4) $\rightarrow$ $0 = \frac{\partial f(u_t, x_t)}{\partial u_t} + \delta \cdot V'(x_{t+1}) \cdot \frac{\partial g(u_t, x_t)}{\partial u_t}$ $\Leftrightarrow$

$$\Leftrightarrow V'(x_{t+1}) = -\frac{1}{\delta} \cdot \frac{\frac{\partial f(u_t, x_t)}{\partial u_t}}{\frac{\partial g(u_t, x_t)}{\partial u_t}} \quad (6) \Leftrightarrow$$

$$\Leftrightarrow V'(x_t) = -\frac{1}{\delta} \cdot \frac{f_u(u_{t-1}, x_{t-1})}{g_u(u_{t-1}, x_{t-1})} \quad (7)$$

Substitute $V'(x_t)$, $V'(x_{t+1})$ in (5) from (6) + (7)

$\downarrow$

Envelope Condition becomes a function of
$(x_{t+1}, x_t, x_{t-1}, u_t, u_{t-1})$ (▲)

u_t: $0 = \frac{\partial f(u_t, x_t)}{\partial u_t} + \delta \cdot V'(x_{t+1}) \cdot \frac{\partial g(u_t, x_t)}{\partial u_t}$ (4)

ENVELOPE CONDITION

$$V'(x_t) = \frac{\partial f(u_t, x_t)}{\partial x_t} + \delta \cdot V'(x_{t+1}) \cdot \left[1 + \frac{\partial g(u_t, x_t)}{\partial x_t}\right] \quad (5)$$

Lecture 1 | A3. Optimal Control ④

(not common in discrete time)

I, II and +

a dynamic system wrt x, u, m

A. Discrete Time

$$\max_{\{u_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \delta^t \cdot f(u_t, x_t) \quad 0 < \delta < 1$$

s.t.

(+) $x_{t+1} - x_t = g(x_t, u_t)$, $t = 0, 1, \ldots$

x_0 given

$x \rightarrow$ vector of STATE variables (predetermined)

$u \rightarrow$ " " CONTROL " (state)

[the (+) refers to the constraints]

Define Hamiltonian

$$\boxed{\mathcal{H}_t \equiv f(u_t, x_t) + m_t \cdot g(x_t, u_t)}$$

or

$$\bar{H} = \delta^t \cdot f(u_t, x_t) + \lambda_t \cdot g(x_t, u_t)$$

F.O.C

Controls: $\boxed{\frac{\partial \mathcal{H}}{\partial u_t} = 0} \Leftrightarrow \frac{\partial f(u_t, x_t)}{\partial u_t} + m_t \cdot \frac{\partial g(u_t, x_t)}{\partial u_t} = 0$ (I)

States: $\boxed{m_t - \delta \cdot \frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = \delta \cdot m_{t+1}}$

where $\frac{\partial \mathcal{H}_{t+1}}{\partial x_{t+1}} = \frac{\partial f(u_{t+1}, x_{t+1})}{\partial x_{t+1}} + m_{t+1} \cdot \frac{\partial g(u_{t+1}, x_{t+1})}{\partial x_{t+1}}$ (II)

So: $m_t = \delta \cdot \left[m_{t+1} \cdot \left(1 + \frac{\partial g(u_{t+1}, x_{t+1})}{\partial x_{t+1}}\right) + \frac{\partial f(u_{t+1}, x_{t+1})}{\partial x_{t+1}} \right]$