LECTURE VII:
The Deterministic Overlapping Generations Model (OLG) and Public Debt

A. Model

1. 1. Preliminaries:

We consider an economy that consists of a large number of individuals that lives for two periods, but each generation of individuals is born in a staggered fashion. In any given period t, a new generation of individuals is born, who live during period t, when they are young, and in period t+1, when they are old. Thus, in any given period t+1, two generations of individuals coexist- the young born in period t+1 and the old born in period t. Further, we assume that only the young are able to work. Thus, an individuals that is born in period t, works in period t and enjoys retirement in period t+1. However, because individuals wish to consume in both periods of their life, to do so they have to save when they are young, so as to finance their consumption when they are old. Furthermore, we assume that individuals cannot leave bequests to the members of future generations and therefore, individuals have no endowment, other than the time available for work. Finally, we assume that labor force grows at a fixed rate, g_n:

$$n_{t+1} = (1+g_n)\, n_t;\ n_0 \text{ given} \tag{VII.1}$$

Production of a single homogeneous final good takes place in a large number of firms, m, using capital and labor services, supplied by individuals. Each individual, when young is endowed with one unit of labor that is supplied inelastically. Individuals buy the single homogeneous final good from firms and they consume it or save it and invest it in the form of physical capital, when they are young. And, use the proceeds from their savings to buy the single homogeneous when they are old. All product and factor of production markets are competitive. And, households own the firms.

2. Production Technology:

In each and every period, t, production of the single homogeneous final good takes place according to the following production technology:

$$Y_t \le F(K_t, z_t L_t) \tag{VII.2}$$

where: Y_t, K_t, L_t is final good output, input of capital services, and input of labor services, respectively. And, z_t is a technological efficiency parameter that represents Harrod-neutral technical progress. The level of technological efficiency, z_t, is assumed to grow at a fixed rate, g_z:

$$z_{t+1} = (1+g_g)\, z_t;\ z_0 \text{ given} \tag{VII.3}$$

The production function $F(K\ ,L\ )$ is assumed to be as in the Neoclassical Growth model (Lecture I), so that the economy's production technology can be expressed in terms of the productivity function:

$$y_t \leq F(k_t, 1) \equiv f(k_t) \qquad \text{(VII.4)}$$

where : y_t *and* k_t are the output and capital input in the economy per "efficient" young individual. That is,

$$y_t = \frac{mY_t}{n_t z_t} \text{ and } k_t = \frac{mK_t}{n_t z_t}$$

And, the productivity function $f(k\ )$ is twice continuously differentiable, strictly increasing and strictly concave and such that: $f(0) = 0$, $f'(k\ ) \to 0$ as $k_t \to \infty$ and $f'(k\ ) \to \infty$ as $k_t \to 0$.

3. <u>Preferences</u> :

Preferences of an individual born in period t are characterized by a time-separable utility function of the form:

$$U_t = u(c_t^y) + \beta u(c_{t+1}^o) \qquad \text{(VII.5)}$$

where: $\beta \in (0,1)$ is a time discount factor, c_t^y and c_{t+1}^o is the consumption of the in individual when she is young in period t and when she is old in period t+1, respectively. $u(\bullet)$ is a period utility function such that: $u(\bullet)$ is twice continuously differentiable, strictly increasing, strictly concave and satisfies the Inada conditions: $u'(\text{c}) \to 0$ as $c \to \infty$ and $u'(\text{c}) \to \infty$ as $c \to 0$.

4. <u>Budget Constraints</u>:

According to what was mentioned above, the budget constraints facing an individual that is born in period t, in each of the two periods that she lives, are:

$$c_t^y + i_t \leq w_t \qquad \text{(VII.6)}$$

$$c_{t+1}^o \leq (1 + r_{t+1}) i_t \qquad \text{(VII.7)}$$

where: i_t is saving/investment in period t, w_t is wage income in period t, d_t is the dividend received from firms in period t and r_{t+1} is the real return on investment in period t+1.

5. Capital Transition Constraint: $k_{t+1} = i_t$

Physical capital depreciates fully in one period, so that:

$$(1+g_n)\ (1+g_z)k_{t+1} = i_t \tag{VII.8}$$

6. Physical Constraints:

$$c_t^y\ , c_{t+1}^o \geq 0 \tag{VII.9}$$

7. Initial Condition:

Finally, we take the economy's initial stock of capital as given:

$$k_o \in [0, \infty) \text{ given} \tag{VII.10}$$

8. Equilibrium:

We are interested in characterizing the equilibrium (resource allocation) of this economy. Competitive equilibrium is defined as a sequence of the form $\{(c_t^y, c_{t+1}^o, i_t, k_{t+1}), (K_t, L_t, Y_t), (r_t, w_t, d_t)\}_{t=0}^{\infty}$, such that:

(I) Given $\{(r_t, w_t, d_t)\}_{t=0}^{\infty}$, $\{(c_t^y, c_{t+1}^o, i_t, k_{t+1})\}_{t=0}^{\infty}$ is a solution to the problems of all individuals born in period 0 and thereafter: That is, $(c_t^y, c_{t+1}^o, i_t, k_{t+1})$ is a solution to :

$$\max_{(c_t^y,\, c_{t+1}^o,\, i_t,\, k_{t+1})} \quad [u(c_t^y) + \beta u(c_{t+1}^o)]$$

subject to: (VII.6)-(VII.7), $\forall t \in \mathbb{N}_+$.

(II) Given $\{(r_t, w_t, d_t)\}_{t=0}^{\infty}$, $\{(K_t, L_t, Y_t)\}_{t=0}^{\infty}$ is a solution to the problems of all firms operating in period 0 and thereafter. That is, (K_t, L_t, Y_t) is a solution to :

$$\max_{(K_t, L_t, Y_t) \geq 0} \quad \pi_t$$

subject to: (VII.1)-(VII.4), $\forall t \in \mathbb{N}_+$,

where: π_t stands for the representative firm's profits:

$$\pi_t = Y_t - r_t K_t - w_t z_t L_t \tag{VII.11}$$

(II) Given $\{(c_t^y, c_{t+1}^o, i_t, k_{t+1}), (K_t, L_t, Y_t)\}_{t=0}^{\infty}$, prices $\{(r_t, w_t, d_t)\}_{t=0}^{\infty}$ clear all markets: That is,

$$\begin{aligned} n_t &= mL_t \\ n_t z_t k_t &= mK_t \\ n_t z_t y_t &= mY_t \end{aligned} \tag{VII.12}$$

where: $y_t = c_t^y + \dfrac{c_t^o}{(1+g_n)(1+g_z)} + i_t$

B. The Workings of the Model

1. Characterization of the Equilibrium:

As in the case of the Neoclassical Growth Model, the solution to the representative firm's problem, provided that $k_o \in (0, \infty)$, is interior (i.e., $(K_t, L_t, Y_t) > 0$. Necessary and sufficient conditions for this, unique solution, are:

$$F_K(K_t, z_t L_t) = r_t \tag{VII.13}$$

$$F_L(K_t, z_t L_t) = w_t \tag{VII.14}$$

And, again as in the Neoclassical Growth Model, in view of the market equilibrium conditions (VII.12), the preceding conditions may be expressed in terms of the productivity function, as follows:

$$f'(k_t) = r_t \tag{VII.15}$$

$$f(k_t) - f'(k_t)k_t = w_t \text{ [1]} \tag{VII.16}$$

Also, it follows that profits and therefore dividends are zero.

Turning, now, to the problem of the individual born in period t, in view of (VII.6) – (V.II.7), the intertemporal budget constraint facing her is given by:

$$c_t^y + \frac{(1+g_n)(1+g_z)}{(1+r_{t+1})} k_{t+1} \leq w_t \tag{VII.17}$$

And, in view of (VII.17), the nonnegativity conditions in (VII.9) can be written as:

$$0 \leq k_{t+1} \leq \frac{(1+r_{t+1})\, w_t}{(1+g_n)(1+g_z)} \tag{VII.18}$$

Clearly, in view of the form of the utility function (i.e., strictly increasing in the consumption levels of the two periods), the budget constraints in (VII.6) and (VII.7) must be statisfied with equality in the solution. Thus, the problem of the individual born in period t can be restated as follows:

$$\max_{k_{t+1}} \; \{u[w_t - (1+g_n)(1+g_z)k_{t+1}] + \beta u[(1+g_n)(1+g_z)(1+r_{t+1})k_{t+1}]\}$$

subject to (VII.18).

Finally, as in the problem of the representative firm, the Inada condition on the period utility function, $u'(\mathrm{c}) \to \infty$ as $c \to 0$, ensures that the solution to the preceding problem will be interior. Hence, the necessary and sufficient condition for the unique solution to the problem of the individual born in period t is:

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1+r_{t+1}) \tag{VII.19}$$

In view of (VII.15) – (VII.16), and unlike in the case of the Neoclassical Growth Model, (VII.19) is a first order difference equation in k_t:

[1] In (VII.16) and thereafter, w_t is

$$u'[f(k_t) - f'(k_t)k_t - (1+g_n)(1+g_z)k_{t+1}) = \beta u'[(1+g_n)(1+g_z)(1+f'(k_{t+1}))k_{t+1}](1+f'(k_{t+1}))$$

(VII.20)

Since this is a first order difference equation in k_t, it can be solved for given k_o, to yield the unique equilibrium path of the economy's capital stock per efficient young individual. Suppose this solution is:

$$k_{t+1} = g(k_t)\,;\; k_o \in (0,\infty)$$

<u>Exercise 1</u>: Show that if $u(c) = \ln c$ and $f(k) = Ak^{\alpha}; A > 0, \alpha \in (0,1)$:

(a) (VII.20) implies:

$$k_{t+1} = \frac{\beta(1-\alpha)A}{(1+\beta)(1+g_n)(1+g_z)} k^{\alpha} \equiv g(k_t) \quad \text{(VII.21)}$$

(b) Show that there exists a $k^* \in (0,\infty)$ such that if $k_0 < k^*$,
$k_0 < k_1 < \ldots\ldots < k_t < k_{t+1} < \ldots\ldots k^*$ and if $k_0 > k^*$,
$k^* < \ldots\ldots < k_{t+1} < k_t < \ldots\ldots k_1 < k_0$.

(c) Show that $k_t \to k^*$ for all $k_0 \in (0,\infty)$

(d) Characterize the effects on k^* brought about by a change in A, α, β, g_n, and g_z.

<u>Exercise 2:</u> Characterize the transitional dynamics and the steady states of the model if $u(c) = \ln c$ and $f(k)$ is "s-shaped".

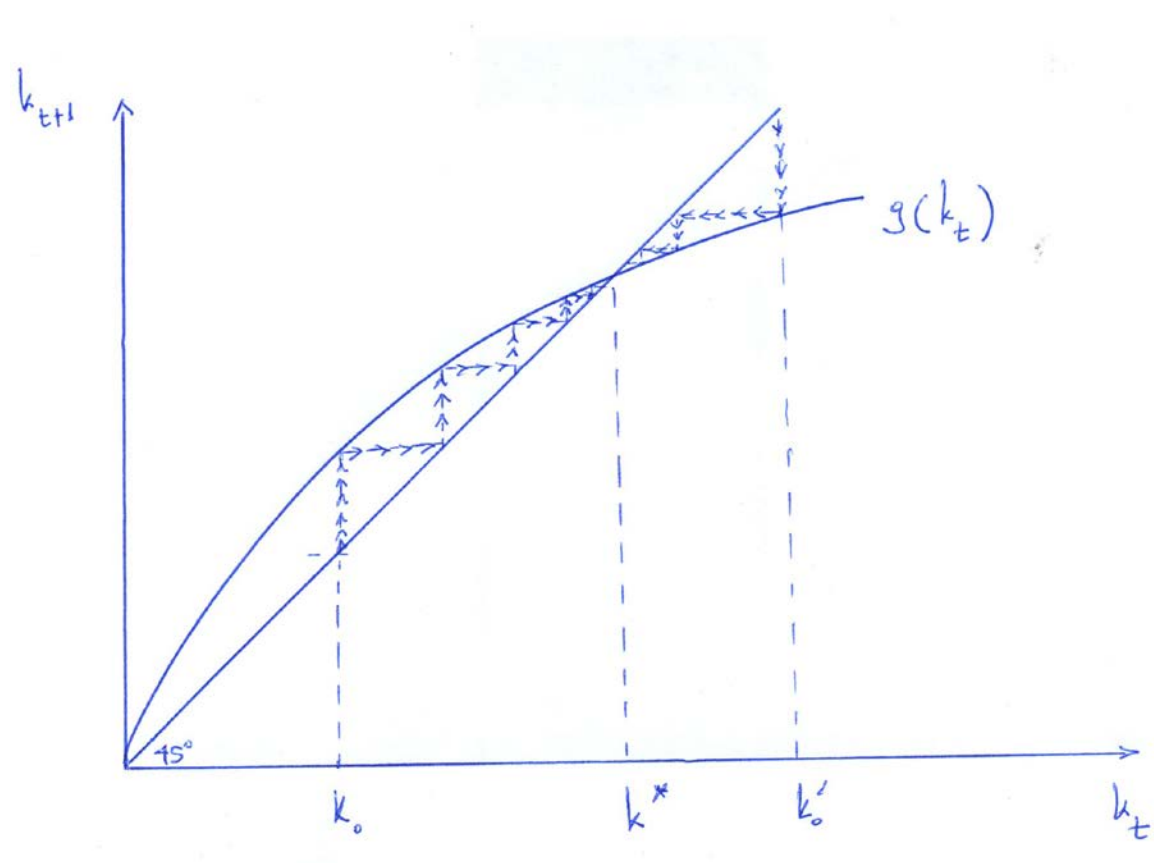

Figure 1: Overlapping Generations Steady State and Transitional Dynamics

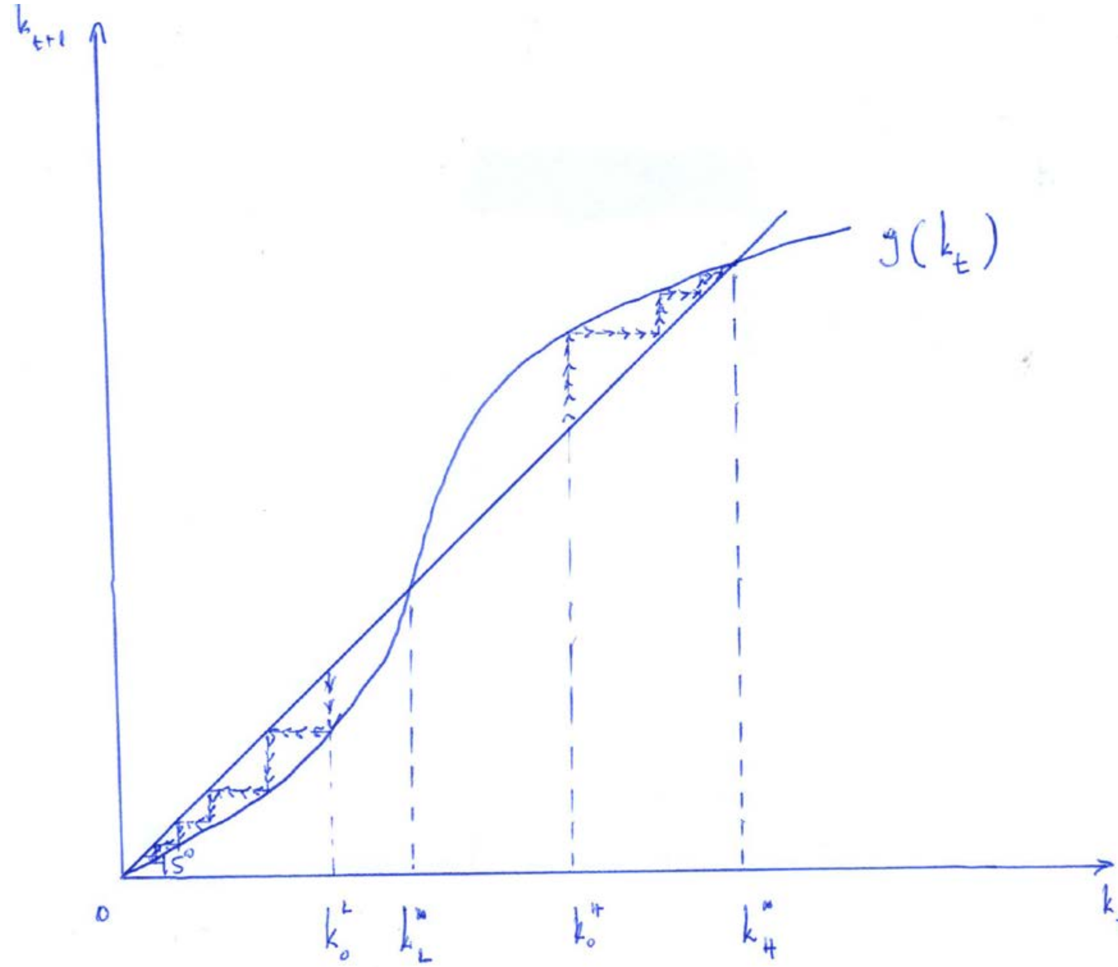

Figure 2: The Case of a Poverty Trap

C. Public Debt

In discussing Ricardian Equivalence in the RBC model, we mentioned that one way this property may not hold is when the agents that buy government bonds and get the associated interest payments are different from those that pay the taxes to cover these interest payments. The OLG model provides a suitable framework to explore this possibility. In the OLG model of the previews subchapters, we introduce a government that taxes those individuals that are currently young to pay interest on its outstanding debt to those that they are currently old. To illustrate the importance of this asymmetry for Ricardian Equivalence, we consider only the case of a single lump sum tax.

1. Government Budget Constraint:

In each and every period t, government faces a budget constraint of the form:

$$B_{t+1} = (1+r_t^b)B_t + G_t - T_t \tag{VII.23}$$

where: B_t is public debt at the beginning of period t, G_t is government expenditures in period t, T_t is government revenues in period t and r_t^b is the rate of interest associated with public debt in period t.

The best way to conceptualize (VII.23) is to think of a situation where at the beginning of any given period, t, government issues one period bonds that promise to pay back, at the end of this period, a fixed principal (i.e., the face value of the bond) and a coupon that is equal to r_t^b times this principal. Then, B_t is the number of bonds issued times the principal. And, we assume that government expenditures are made on pure public good, that we need specify no further. Dividing both hand sides of (VII,23) by $n_t z_t$, we can re-express the government budget constraint as:

$$(1+g_n)(1+g_z)b_{t+1} = (1+r_t^b)b_t + g_t - \tau_t \tag{VII.24}$$

where: $b_t, g_t,$ and τ_t stand for public debt, government expenditures, and government revenues, respectively, per efficient young individual, born in period t.

Solving equation (VII.24) forward gives:

$$b_{t+T} = \left[\prod_{j=0}^{T-1} \frac{(1+r_{t+j}^b)}{(1+g_n)(1+g_z)}\right] b_t + \sum_{j=0}^{T-2}\left[\prod_{i=j}^{T-1} \frac{(1+r_{t+i}^b)}{(1+g_n)(1+g_z)}\right] \frac{(g_{t+j}-\tau_{t+j})}{(1+g_n)(1+g_z)} + \frac{(g_{t+T-1}-\tau_{t+T-1})}{(1+g_n)(1+g_z)} \tag{VII.25}$$

And, solving for b_t, (VII.25) gives:

$$b_t = \left[\prod_{j=0}^{T-1} \frac{(1+g_n)(1+g_z)}{(1+r_{t+j}^b)}\right] b_{t+T} + \sum_{j=0}^{T-1}\left[\prod_{i=0}^{j} \frac{(1+g_n)(1+g_z)}{(1+r_{t+j}^b)}\right] \frac{(\tau_{t+j}-g_{t+j})}{(1+g_n)(1+g_z)} \tag{VII.26}$$

We impose the No Ponzi game condition:

$$\left[\prod_{j=0}^{T-1}\frac{(1+g_n)(1+g_z)}{(1+r^b_{t+j})}\right]b_{t+T} \to 0 \; as \; T \to \infty \qquad \text{(VII.27)}$$

Recall, that the implication of this condition is that the government cannot permanently recycle its debt. That is, issue new debt that covers its total deficits, forever. In view of (VII.27), the level of existing debt (i.e., old debt) in any given period must be equal to the discounted future stream of all future primary surpluses:

$$b_t = \sum_{j=0}^{\infty}\left[\prod_{i=0}^{j}\frac{(1+g_n)(1+g_z)}{(1+r^b_{t+j})}\right]\frac{(\tau_{t+j}-g_{t+j})}{(1+g_n)(1+g_z)} \qquad \text{(VII.28)}$$

At this point it will be useful to investigate the implications of (VII.28) for the steady state of debt. Suppose that: $r^b_t = r^b$ and $\tau_t - g_t = \tau - g$. Then, (VII.28) implies that:

$$b = \frac{(\tau - g)}{(1+r^b)-(1+g_n)(1+g_z)} \simeq \frac{\tau - g}{r^b - g_n - g_z} \qquad \text{(VII.29)}$$

It follows, that b =0 if and only if $\tau - g$ =0 and b >0 if either $\tau - g$ >0 and $r^b > g_n + g_z$, or $\tau - g$ <0 and $r^b < g_n + g_z$. That is, for strictly positive steady state debt per efficient young individual, either the government must run primary surpluses and the interest rate of debt must be greater than the economy's growth rate (this will be shown later), or the government must run primary deficits, in which case the interest rate of debt must be greater than the economy's growth rate.

2. Individual Budget Constraints:

According to what was mentioned above, the budget constraints facing an individual that is born in period t, in each of the two periods that she lives, are:

$$c^y_t + (1+g_n)(1+g_z)(k_{t+1}+b_{t+1}) \leq w_t - \tau_t \qquad \text{(VII.30)}$$

$$c^o_{t+1} \leq (1+g_n)(1+g_z)[(1+r_{t+1})k_{t+1} + (1+r^b_{t+1})b_{t+1})] \qquad \text{(VII.31)}$$

These constraints incorporate the assumptions that a lump sum tax is levied on the young and that the young can now invest in physical capital as well as government bonds.

3. Equilibrium:

All market equilibrium conditions, given $\{g_t, \tau_t\}_{t=0}^{\infty}$, remain the same, except the product market equilibrium, where, $y_t = c_t^y + \dfrac{c_t^o}{(1+g_n)(1+g_z)} + i_t + \; g_t$.

4. <u>Characterization of Equilibrium:</u>

In view of these budget constraints, it follows as in Subsection B, that the problem of the individual born in period t can be stated as follows

$$\max_{k_{t+1},\, b_{t+1}} \left(u[w_t - \tau_t - (1+g_n)(1+g_z)(k_{t+1}+b_{t+1})] + \beta u\{(1+g_n)(1+g_z)[(1+r_{t+1})k_{t+1} + (1+r_{t+1}^b)b_{t+1}]\} \right)$$

The necessary and sufficient conditions, for the unique solution to this problem, are:

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1+r_{t+1}) \qquad \text{(VII.32)}$$

$$u'(c_t^y) = \beta u'(c_{t+1}^o)(1+r_{t+1}^b) \qquad \text{(VII.33)}$$

The obvious implication of these two conditions is:

$$r_{t+1}^b = r_{t+1} = f'(k_{t+1}) \qquad \text{(VII.34)}$$

In view of (VII.34), (VII.32) and the law of notion of debt per efficient young individual can be written as follows:

$$\begin{aligned} &u'[f(k_t) - f'(k_t)k_t - \tau_t - (1+g_n)(1+g_z)(k_{t+1}+b_{t+1})] = \\ &\beta u'\{(1+g_n)(1+g_z)[1+f'(k_{t+1})](k_{t+1}+b_{t+1})\}(1+f'(k_{t+1})) \end{aligned} \qquad \text{(VII.35)}$$

$$(1+g_n)(1+g_z)b_{t+1} = [1+f'(k_{t+1})]b_t + g_t - \tau_t \qquad \text{(VII.36)}$$

Note that (VII.35)- (VII.36) form a system two first order difference equations in k_{t+1}, b_{t+1}. The solution of this system is of the form:

$$k_{t+1} = g^k(k_t, b_t; g_t, \tau_t)$$

$$b_{t+1} = g^b(k_t, b_t; g_t, \tau_t)$$

$$(k_o, b_0) \in (0,\infty) \text{ x}[0,\infty) \text{ given}$$

Clearly, debt affects the equilibrium of the economy. Unlike the Neoclassical Growth model, debt is no longer is neither indeterminate nor irrelevant. The system, however may have none, one or multiple steady states.

5. An Example

Suppose that: $u(c) = \ln c$, $f(k) = Ak^\alpha; A > 0, \alpha \in (0,1)$. Following these specifications, (VII.35) and (VII.36) yield:

$$k_{t+1} = \frac{\beta(1-\alpha)Ak^\alpha}{(1+\beta)(1+g_n)(1+g_z)} - b_{t+1} - \frac{\beta\tau_t}{(1+\beta)} \qquad \text{(VII.37)}$$

$$b_{t+1} = \frac{(1+\alpha Ak_t^{\alpha-1})b_t}{(1+g_n)(1+g_z)} + \frac{g_t}{(1+g_n)(1+g_z)} - \frac{\tau_t}{(1+g_n)(1+g_z)} \qquad \text{(VII.38)}$$

To investigate the properties of this model, suppose that g_t, τ_t adjust, so that:

$$(1+\alpha Ak^{\alpha-1})b_t + g_t - \frac{\tau_t}{(1+\beta)} = \overline{b} > 0$$

In view of (VII,38), we may think of $\overline{b}$ as a strictly increasing function of debt at the beginning of next period.

Then, (VII.37) reduces to:

$$k_{t+1} = \frac{\beta(1-\alpha)Ak^\alpha}{(1+\beta)(1+g_n)(1+g_z)} - \overline{b} = g(k_t) - \overline{b} \qquad \text{(VII.39)}$$

where $g(\bullet)$ is defined in (VII.21). The properties of $g(k_t) - \overline{b}$ are illustrated in Figure 3. Clearly, $k_{t+1} = k_t$ if k_t is equal to k_1^* or k_2^* and $0 < k_1^* < k_2^* < k^*$, where k^* is the equilibrium without public debt in Figure 1, above . Moreover, $k_{t+1} > k_t$ if $k_1^* < k_t < k_2^*$ and $k_{t+1} < k_t$, if $k_1^* > k_t$ or $k_2^* < k_t$. Thus, k_1^* is an unstable and k_2^* is a stable equilibrium. Note then, that an increase in the constant debt level (i.e., $\bar{b}$) lowers the stable steady state value of capital and the transition trajectory towards the new steady state. Essentially, this is like a transfer to the currently old that causes there consumption to increase, but this increase in debt causes the taxes of the future generations to increase, lowering their consumption and investment.

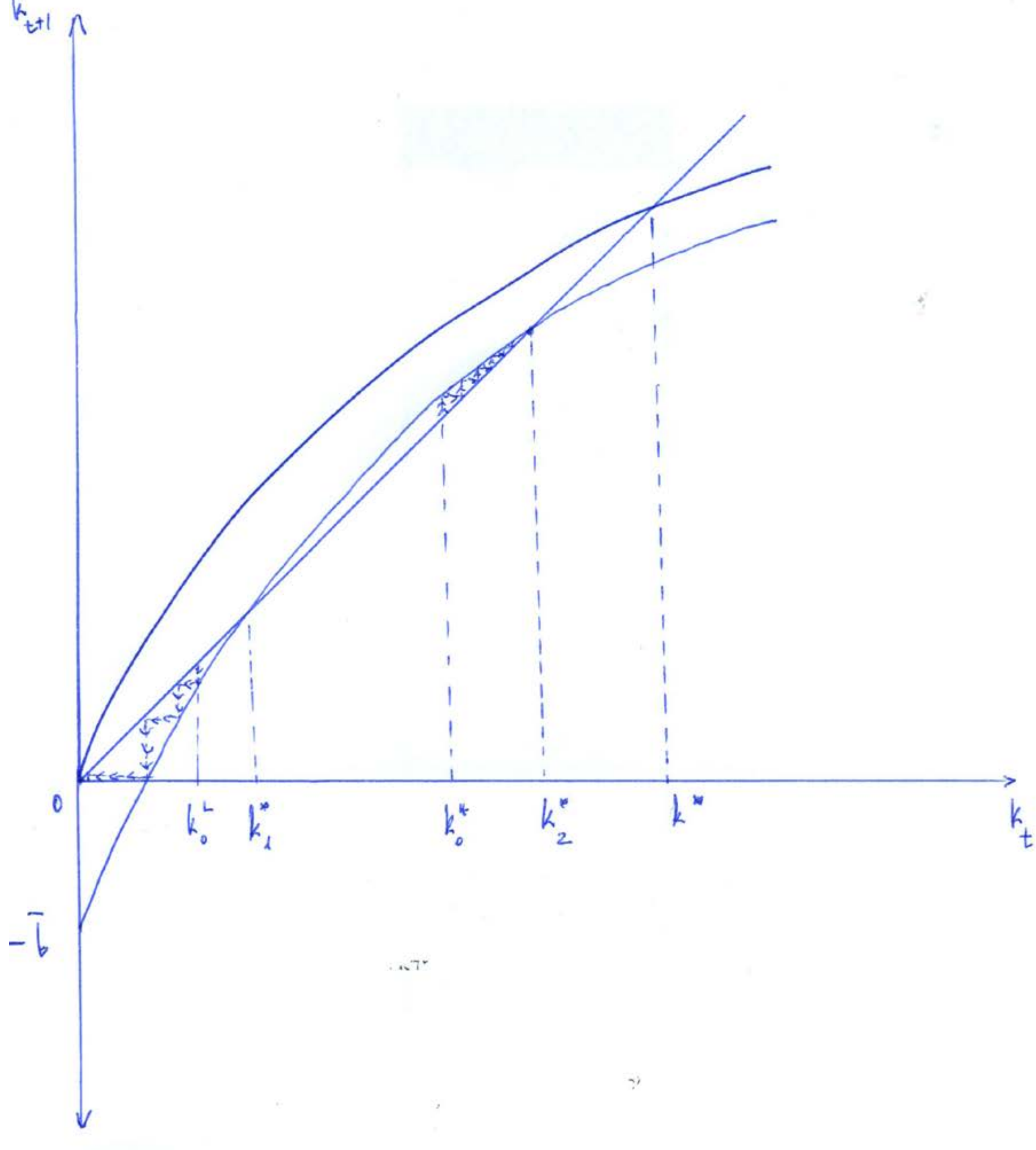

Figure 3 : Overlapping Generations Steady State and Transitional Dynamics with Public Debt