LECTURE V :

The New Keynesian Model

(Based on the lecture notes of Fabrice Collard)

A note on Lagrange multipliers

Lagrange multipliers are auxiliary state variables that help us characterize the solution to optimization problems.

In the proof of the necessary condition for an interior solution to the Basic Stochastic Dynamic Problem we showed that if $\{x^*_{t+1}(\theta^t)\}_{t=0}^{\infty}$ is a solution to this problem $x^*_{t+1}(\theta^t)$ must be a solution to:

$$\max_{x_{t+1}(\theta^t)} \{ \Phi[x^*_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + \beta E_t \Phi[x_{t+1}(\theta^t), x^*_{t+2}(\theta^{t+1}), \theta_{t+1}] \} \quad (1)$$

$$\text{s.t.} \quad x_{t+1}(\theta^t) \in \Gamma[x^*_t(\theta^{t-1}), \theta_t] \quad (2)$$

$$x^*_{t+2}(\theta^{t+1}) \in \Gamma[x_{t+1}(\theta^t), \theta_{t+1}] \quad (3)$$

for all $t \in \mathbb{N}_+$

Suppose that the constraint correspondence $\Gamma(x,\theta)$ is defined in terms of inequality constraints of the form: (i.e., $x_{t+1} \in \{\mathbb{R}^n \mid g(x,y,\theta) \leq 0\}$)

$$g(x, y, \theta) \leq 0 \quad (4)$$

where $g: \mathbb{R}^n \times \mathbb{R}^n \times \Theta \longrightarrow \mathbb{R}^m$ is a differentiable function. Then, it follows from standard nonlinear optimization theory that there exists vectors $\Lambda_t \equiv \Lambda(\theta^t) : \Theta^t \longrightarrow \mathbb{R}^m_+$ such that, the solution to (1)-(3) is equivalent to the solution to:

$$\max_{x_{t+1}(\theta^t)} \Big\{ \Phi [x_t^*(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + b\, E_t \Phi [x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}] + \Lambda(\theta^t)' g [x_t^*(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + b E_t \Lambda(\theta^{t+1})' g [x_{t+1}(\theta^t), x_{t+2}^*(\theta^{t+1}), \theta_{t+1}] \Big\} \quad (5)$$

Moreover,

$\Lambda(\theta^t)\, g [x_t^*(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \geq 0$ and the jth component of $\Lambda(\theta^t)$ is zero if and only if the j^{th} component of $g [x_t^*(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]$ is less than zero (6)

$b E_t \Lambda(\theta^{t+1})\, g [x_{t+1}(\theta^t), x_{t+2}^*(\theta^{t+1}), \theta_{t+1}] \geq 0$ and the j^{th} component of $\Lambda(\theta^{t+1})$ is zero if and only if the j^{th} component of $g [x_{t+1}(\theta^t), x_{t+2}^*(\theta^{t+1}), \theta_{t+1}]$ is less than zero. (7)

Clearly, then, the first order conditions associated with (5) is:

$$\Phi_2 [x_t^*(\theta^{t+1}), x_{t+1}^*(\theta^t), \theta_t] + b\, E_t \Phi [x_{t+1}^*(\theta^t), x_{t+2}^*(\theta^{t+1}), \theta_{t+1}] + \Lambda(\theta^t)\, g_2 [x_t^*(\theta^{t-1}), x_{t+1}^*(\theta^t), \theta_t] + b E_t \Lambda(\theta^{t+1})\, g_1 [x_{t+1}^*(\theta^t), x_{t+2}^*(\theta^{t+1}), t+1] = 0 \quad (8)$$

Note g_1, g_2 are $m \times n$ matrices and the LHS of (8) is a $1 \times n$ vector.

The latter equation is the stochastic Euler condition augmented with Lagrange multipliers. Typically, to solve it, one would need the auxiliary (complementary slackness) conditions (6) and (7). Further, it is clear that the j^{th} element of the $\Lambda_t(\theta^t)$ vector may be interpreted as the marginal increment of the objective function associated with relaxing the constraint associated with the j^{th} element of g.

Finally, note that Λ_t is a function of θ^t. Thus, $\Lambda_t(\theta^t)$ is itself a random variable. Finally, conditions (6) - (8) are also sufficient when Φ and g are concave functions and they are necessary and sufficient if Φ and g are strictly concave.

1 The model

The set up is standard. The economy is populated by a large number of identical infinitely–lived households and economy consists of two sectors: one producing intermediate goods and the other final goods. The intermediate good is produced with capital and labor and the final good with intermediate goods. The final good is homogeneous and can be used for consumption (private and public) and investment purposes.

1.1 The Household

Household preferences are characterized by the lifetime utility function:[1]

$$E_t \sum_{\tau=0}^{\infty} \beta^{\tau} U\left(c_{t+\tau}, \frac{M_{t+\tau}}{P_{t+\tau}}, \ell_{t+\tau}\right) \tag{1}$$

where $0 < \beta < 1$ is a constant discount factor, c denotes consumption , M/P real balances and ℓ leisure. The utility function,$U\left(c, \frac{M}{P}, \ell\right) : \mathbb{R}_{+} \times \mathbb{R}_{+} \times [0,1] \longrightarrow \mathbb{R}$ is increasing and concave in its arguments.

The household is subject to the following time constraint

$$\ell_t + h_t = 1 \tag{2}$$

where h denotes hours worked. The total time endowment is normalized to unity.

In each and every period, the representative household faces a budget constraint of the form

$$B_t + M_t + P_t(c_t + x_t + \tau_t) \quad \leq \quad R_{t-1}B_{t-1} + M_{t-1} + N_t + \Pi_t + P_t w_t h_t + P_t z_t k_t \tag{3}$$

where B_t are M_t are nominal bonds and money acquired during period t, P_t is the nominal price of the final good, R_{t-1} is the nominal interest rate, w_t and z_t are the real wage rate and real rental rate of capital. The household owns k_t units of physical capital, makes an additional investment of x_t, consumes c_t and supplies h_t units of labor. It pays lump sum taxes τ_t, receives a transfer of money N_t from the government and finally claims the profits, Π_t, earned by the firms.

Capital accumulates according to the law of motion

$$k_{t+1} = x_t + (1-\delta)k_t \tag{4}$$

$\delta \in [0,1]$ denotes the rate of depreciation.

[1] $E_t(.)$ denotes mathematical conditional expectations. Expectations are conditional on information available at the beginning of period t.

The first order conditions lead to the following money demand equation

$$
\begin{aligned}
(c) \qquad & U_c(t) = P_t \Lambda_t && (5)\\
(\ell) \qquad & U_\ell(t) = P_t \Lambda_t w_t && (6)\\
(M) \qquad & \Lambda_t = \frac{U_m(t)}{P_t} + \beta E_t \Lambda_{t+1} && (7)\\
(k) \qquad & \lambda_t = \beta E_t \left[\Lambda_{t+1} P_{t+1}(z_{t+1} + 1 - \delta)\right] && (8)\\
(B) \qquad & \Lambda_t = \beta R_t E_t \Lambda_{t+1} && (9)
\end{aligned}
$$

where Λ_t denotes the Lagrange multiplier associated to the budget constraint.

1.2 Final sector

The final good is produced by combining intermediate goods. This process is described by the following CES function

$$y_t = \left(\int_0^1 y_t(i)^\theta \mathrm{d}i\right)^{\frac{1}{\theta}} \tag{10}$$

where $\theta \in (-\infty, 1)$. θ determines the elasticity of substitution between the various inputs. The producers in this sector are assumed to behave competitively and to determine their demand for each good, $y_t(i)$, $i \in (0,1)$ by maximizing the static profit equation

$$\max_{\{X_t(i)\}_{i\in(0,1)}} P_t y_t - \int_0^1 P_t(i) y_t(i) \mathrm{d}i \tag{11}$$

subject to (10), where $P_t(i)$ denotes the price of intermediate good i. This yields demand functions of the form:

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\theta-1}} y_t \tag{12}$$

and the following general price index

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\theta}{\theta-1}} \mathrm{d}i\right)^{\frac{\theta-1}{\theta}} \tag{13}$$

The final good may be used for consumption — private or public — and investment purposes.

1.3 Intermediate goods producers

Each firm i, $i \in (0,1)$, produces an intermediate good by means of capital and labor according to a constant returns–to–scale technology, represented by the production function

$$y_t(i) = a_t k_t(i)^\alpha h_t(i)^{1-\alpha} \text{ with } \alpha \in (0,1) \tag{14}$$

where $k_t(i)$ and $h_t(i)$ respectively denote the physical capital and the labor input used by firm i in the production process. a_t is an exogenous stationary stochastic technology shock. Assuming that each firm i operates under perfect competition in the input markets, the firm determines its production plan so as to minimize its total cost

$$\min_{\{k_t(i),h_t(i)\}} P_t w_t h_t(i) + P_t z_t k_t(i)$$

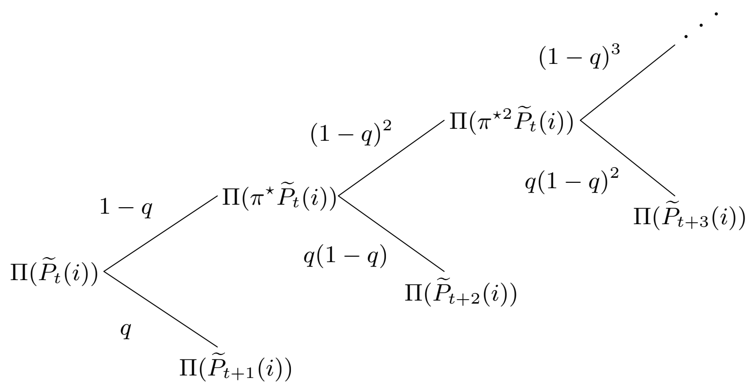
subject to (14). This yields to the following expression for total costs:

$$P_t s_t y_t(i)$$

where the real marginal cost, s, is given by $\frac{w_t^{1-\alpha} z_t^{\alpha}}{\Delta a_t}$ with $\Delta = \alpha^{\alpha}(1-\alpha)^{1-\alpha}$

Intermediate goods producers are monopolistically competitive, and therefore set prices for the good they produce. We follow Calvo [1983] in assuming that firms set their prices for a stochastic number of periods. In each and every period, a firm either gets the chance to adjust its price (an event occurring with probability q) or it does not. When the firm

Figure 1: Price adjustment scheme



does not reset its price, it just applies steady state inflation to the price it charged in the last period such that $P_t(i) = \pi^{\star} P_{t-1}(i)$. When it gets a chance to do it, firm i resets its price, $\widetilde{P}_t(i)$, in period t in order to maximize its expected discounted profit flow this new price will generate. In period t, the profit is given by $\Pi(\widetilde{P}_t(i))$. In period $t+1$, either the firm resets its price, such that it will get $\Pi(\widetilde{P}_{t+1}(i))$ with probability q, or it does not and its $t+1$ profit will be $\Pi(\pi^{\star}\widetilde{P}_t(i))$ with probability $(1-q)$. Likewise in $t+2$. Figure 1 summarizes all possibilities, such that the expected profit flow generated by setting $\widetilde{P}_t(i)$ in period t writes

$$\max_{\widetilde{P}_t(i)} E_t \sum_{\tau=0}^{\infty} \Phi_{t+\tau}(1-q)^{\tau-1}\Pi(\pi^{\star\tau}\widetilde{P}_t(i))$$

subject to the total demand it faces:

$$y_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\theta-1}} y_t$$

and where $\Pi(\pi^{\star}\tau\widetilde{P}_{t+\tau}(i)) = \left(\pi^{\star\tau}\widetilde{P}_t(i) - P_{t+\tau}s_{t+\tau}\right) y_{t+\tau}(i)$. $\Phi_{t+\tau}$ is an appropriate discount factor related to the way the household value future as opposed to current consumption, such that

$$\Phi_{t+\tau} \propto \beta^{\tau}\frac{\Lambda_{t+\tau}}{\Lambda_t}$$

This leads to the price setting equation

$$\theta\widetilde{P}_t(i)E_t\sum_{\tau=0}^{\infty}(\beta\pi^{\star}(1-q))^{\tau}\Lambda_{t+\tau}\left(\frac{\pi^{\star\tau}\widetilde{P}_t(i)}{P_{t+\tau}}\right)^{\frac{1}{\theta-1}} y_{t+\tau} = E_t\sum_{\tau=0}^{\infty}(\beta(1-q))^{\tau}\Lambda_{t+\tau}\left(\frac{\pi^{\star\tau}\widetilde{P}_t(i)}{P_{t+\tau}}\right)^{\frac{1}{\theta-1}} P_{t+\tau}s_{t+\tau}y_{t+\tau}$$

from which it shall be clear that all firms that reset their price in period t set it at the same level ($\widetilde{P}_t(i) = \widetilde{P}_t$, for all $i \in (0,1)$). This implies that

$$\widetilde{P}_t = \frac{P_t^{\mathrm{N}}}{\theta P_t^{\mathrm{D}}} \tag{15}$$

where

$$P_t^{\mathrm{N}} = E_t\sum_{\tau=0}^{\infty}(\beta(1-q))^{\tau}\Lambda_{t+\tau}\left(\frac{\pi^{\star\tau}\widetilde{P}_t}{P_{t+\tau}}\right)^{\frac{1}{\theta-1}} P_{t+\tau}s_{t+\tau}y_{t+\tau}$$

and

$$P_t^{\mathrm{D}} = E_t\sum_{\tau=0}^{\infty}(\beta\pi^{\star}(1-q))^{\tau}\Lambda_{t+\tau}\left(\frac{\pi^{\star\tau}\widetilde{P}_t}{P_{t+\tau}}\right)^{\frac{1}{\theta-1}} y_{t+\tau}$$

Fortunately, both P_t^{N} and P_t^{D} admit a recursive representation, such that

$$P_t^{\mathrm{N}} = \Lambda_t\left(\frac{\widetilde{P}_t}{P_t}\right)^{\frac{1}{\theta-1}} P_t s_t y_t + \beta(1-q)E_t P_{t+1}^{\mathrm{N}} \tag{16}$$

$$P_t^{\mathrm{D}} = \Lambda_t\left(\frac{\widetilde{P}_t}{P_t}\right)^{\frac{1}{\theta-1}} y_t + \beta\pi^{\star}(1-q)E_t P_{t+1}^{\mathrm{D}} \tag{17}$$

Recall now that the price index is given by

$$P_t = \left(\int_0^1 P_t(i)^{\frac{\theta}{\theta-1}}\mathrm{d}i\right)^{\frac{\theta-1}{\theta}}$$

In fact it is composed of surviving contracts and newly set prices. Given that in each an every period a price contract has a probability q of ending, the probability that a contract signed in period $t-j$ survives until period t and ends at the end of period t is given by $q(1-q)^j$. Therefore, the aggregate price level may be expressed as the average of all surviving contracts

$$P_t = \left(\sum_{j=0}^{\infty} q(1-q)^j(\pi^{\star j}\widetilde{P}_{t-j})^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}}$$

which can be expressed recursively as

$$P_t = \left(q\widetilde{P}_t^{\frac{\theta}{\theta-1}} + (1-q)(\pi^\star P_{t-1})^{\frac{\theta}{\theta-1}}\right)^{\frac{\theta-1}{\theta}} \tag{18}$$

1.4 The monetary authorities

Money is exogenously supplied by the central bank according to the following money growth rule:

$$M_t = \mu_t M_{t-1} \tag{19}$$

where $\mu_t \geqslant 1$ is the exogenous gross rate of growth of money, such that $N_t = M_t - M_{t-1} = (\mu_t - 1)M_{t-1}$. μ_t will be assumed to be an exogenous stochastic process.

1.5 The government

The government finances government expenditure on the domestic final good using lump sum taxes ($g_t = \tau_t$). The stationary component of government expenditures is assumed to follow an exogenous stochastic process, whose properties will be defined later.

1.6 The Equilibrium

An equilibrium of this economy is a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty} = \{w_t, z_t, P_t, R_t, \widetilde{P}_t\}_{t=0}^{\infty}$ and a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty} = \{\{\mathcal{Q}_t^H\}_{t=0}^{\infty}, \{\mathcal{Q}_t^F\}_{t=0}^{\infty}\}$ with

$$\begin{aligned}\{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{c_t, x_t, B_t, k_{t+1}, h_t, M_t\}\\ \{\mathcal{Q}_t^H\}_{t=0}^{\infty} &= \{y_t, y_t(i), k_t(i), h_t(i); i \in (0,1)\}_{t=0}^{\infty}\end{aligned}$$

such that:

(i) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^H\}_{t=0}^{\infty}$ is a solution to the representative household's problem;

(ii) given a sequence of prices $\{\mathcal{P}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{Q}_t^F\}_{t=0}^{\infty}$ is a solution to the representative firms' problem;

(iii) given a sequence of quantities $\{\mathcal{Q}_t\}_{t=0}^{\infty}$ and a sequence of shocks, $\{\mathcal{P}_t\}_{t=0}^{\infty}$ clears the markets

(iv) Prices are set satisfy (15) and (18).

Simple as it seems, there are actually some issues concerning aggregation which should be considered. First of all, *how should we define an aggregate economy?* Let us first recall that from the optimal intermediate good producer's programm, we get

$$\frac{k_t(i)}{h_t(i)} = \frac{\alpha}{1-\alpha}\frac{w_t}{z_t} \Longleftrightarrow \frac{k_t(i)}{h_t(i)} = \frac{k_t}{h_t}\ \forall i \in (0,1)$$

where

$$k_t = \int_0^1 k_t(i)\mathrm{d}i \text{ and } h_t = \int_0^1 h_t(i)\mathrm{d}i$$

The production function for a given intermediate good producer therefore delivers

$$y_t(i) = a_t\left(\frac{k_t}{h_t}\right)^{\alpha} h_t(i)$$

which implies that

$$\int_0^1 y_t(i)\mathrm{d}i = a_t k_t^{\alpha} h_t^{1-\alpha}$$

However, the quantity $\int_0^1 y_t(i)\mathrm{d}i$ differs from y_t, the definition of which involving the elasticity of substitution between goods. Therefore, plugging the expression for the demand for an individual intermediate good in $\int_0^1 y_t(i)\mathrm{d}i$, we get

$$\int_0^1 y_t(i)\mathrm{d}i = \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{\frac{1}{\theta-1}} y_t\mathrm{d}i = \int_0^1 P_t(i)^{\frac{1}{\theta-1}}\mathrm{d}i P_t^{\frac{1}{1-\theta}} y_t$$

We therefore have this component

$$\int_0^1 P_t(i)^{\frac{1}{\theta-1}}\mathrm{d}i$$

which differs from the aggregate price level (see (13)). We therefore define

$$\overline{P}_t = \left(\int_0^1 P_t(i)^{\frac{1}{\theta-1}}\mathrm{d}i\right)^{\theta-1}$$

Note that just as the aggregate price level, and for the same reasons, this price aggregate admits a recursive formulation

$$\overline{P}_t = \left(q\widetilde{P}_t^{\frac{1}{\theta-1}} + (1-q)(\pi^\star \overline{P}_{t-1})^{\frac{1}{\theta-1}}\right)^{\theta-1}$$

Hence, we get

$$\int_0^1 y_t(i)\mathrm{d}i = \left(\frac{\overline{P}_t}{P_t}\right)^{\frac{1}{\theta-1}} y_t = a_t k_t^\alpha h_t^{1-\alpha}$$

which relates the aggregate level of the final good to an aggregate technology through a relative price. The good market clearing condition simply writes as

$$y_t = c_t + x_t + g_t$$

2 Solving the model

2.1 Deflating the economy

So far, we assumed no growth in this economy. However, nominal variables grow at the rate $\overline{\mu}$. We therefore need to deflate all nominal variables. For convenience, we use P_t as a deflator, implying that

$$\lambda_t = \Lambda_t P_t, p_t^{\mathrm{D}} = P_t P_t^{\mathrm{D}}, \widetilde{p}_t = \frac{\widetilde{P}_t}{P_t}, \overline{p}_t = \frac{\overline{P}_t}{P_t}, m_t = \frac{M_t}{P_t},$$

for notational convenience, we will now use the lowercase for $p_t^{\mathrm{N}} = P_t^{\mathrm{N}}$. Finally, we note $\pi_t = P_t/P_{t-1}$. Collecting all deflated equations, we end up with the system:

$$U_c(t) = \lambda_t \tag{20}$$
$$U_\ell(t) = \lambda_t w_t \tag{21}$$
$$\frac{U_m(t)}{U_c(t)} = \frac{R_t - 1}{R_t} \tag{22}$$
$$\lambda_t = \beta E_t\left[\lambda_{t+1}(z_{t+1} + 1 - \delta)\right] \tag{23}$$
$$\lambda_t = \beta R_t E_t \frac{\lambda_{t+1}}{\pi_{t+1}} \tag{24}$$
$$h_t + \ell_t = 1 \tag{25}$$
$$y_t = c_t + x_t + g_t \tag{26}$$
$$\overline{p}_t^{\frac{1}{\theta-1}} y_t = a_t k_t^\alpha h_t^{1-\alpha} \tag{27}$$
$$w_t = (1-\alpha) s_t a_t (k_t/h_t)^\alpha \tag{28}$$

$$z_t = \alpha s_t a_t (k_t/h_t)^{\alpha-1} \tag{29}$$

$$k_{t+1} = x_t + (1-\delta)k_t \tag{30}$$

$$m_t = \mu_t m_{t-1}/\pi_t \tag{31}$$

$$\widetilde{p}_t = \frac{p_t^{\mathrm{N}}}{\theta p_t^{\mathrm{D}}} \tag{32}$$

$$p_t^{\mathrm{N}} = \lambda_t \widetilde{p}_t^{\frac{1}{\theta-1}} s_t y_t + \beta(1-q)E_t p_{t+1}^{\mathrm{N}} \tag{33}$$

$$p_t^{\mathrm{D}} = \lambda_t \widetilde{p}_t^{\frac{1}{\theta-1}} y_t + \beta\pi^\star(1-q)E_t \frac{p_{t+1}^{\mathrm{D}}}{\pi_{t+1}} \tag{34}$$

$$1 = \left(q\widetilde{p}_t^{\frac{\theta}{\theta-1}} + (1-q)\left(\frac{\pi^\star}{\pi_t}\right)^{\frac{\theta}{\theta-1}} \right)^{\frac{\theta-1}{\theta}} \tag{35}$$

$$\overline{p}_t = \left(q\widetilde{p}_t^{\frac{1}{\theta-1}} + (1-q)\left(\frac{\pi^\star}{\pi_t}\overline{p}_{t-1}\right)^{\frac{1}{\theta-1}} \right)^{\theta-1} \tag{36}$$

The log–linear representation of the system is given by

$$\zeta_{cc}\widehat{c}_t + \zeta_{cm}\widehat{m}_t + \zeta_{c\ell}\widehat{\ell}_t = \widehat{\lambda}_t \tag{37}$$

$$\zeta_{\ell c}\widehat{c}_t + \zeta_{\ell m}\widehat{m}_t + \zeta_{\ell\ell}\widehat{\ell}_t = \widehat{\lambda}_t + \widehat{w}_t \tag{38}$$

$$(R^\star-1)(\zeta_{mc}-\zeta_{cc})\widehat{c}_t + (R^\star-1)(\zeta_{mm}-\zeta_{cm})\widehat{m}_t + (R^\star-1)(\zeta_{m\ell}-\zeta_{c\ell})\widehat{\ell}_t = \widehat{R}_t \tag{39}$$

$$\widehat{\lambda}_t = E_t\widehat{\lambda}_{t+1} + (1-\beta(1-\delta))E_t\widehat{z}_{t+1} \tag{40}$$

$$\widehat{\lambda}_t = \widehat{R}_t + E_t\widehat{\lambda}_{t+1} - E_t\widehat{\pi}_{t+1} \tag{41}$$

$$h^\star\widehat{h}_t + (1-h^\star)\widehat{\ell}_t = 0 \tag{42}$$

$$\widehat{y}_t = \frac{c^\star}{y^\star}\widehat{c}_t + \frac{x^\star}{y^\star}\widehat{x}_t + \frac{g^\star}{y^\star}\widehat{g}_t \tag{43}$$

$$\frac{\widehat{\overline{p}}_t}{\theta-1} + \widehat{y}_t = \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{h}_t \tag{44}$$

$$\widehat{w}_t = \widehat{s}_t + \widehat{a}_t + \alpha\widehat{k}_t - \alpha\widehat{h}_t \tag{45}$$

$$\widehat{z}_t = \widehat{s}_t + \widehat{a}_t + (1-\alpha)\widehat{h}_t - (1-\alpha)\widehat{k}_t \tag{46}$$

$$\widehat{k}_{t+1} = \delta\widehat{x}_t + (1-\delta)\widehat{k}_t \tag{47}$$

$$\widehat{m}_t = \widehat{\mu}_t + \widehat{m}_{t-1} - \widehat{\pi}_t \tag{48}$$

$$\widehat{\widetilde{p}}_t = \widehat{p}_t^{\mathrm{N}} - \widehat{p}_t^{\mathrm{D}} \tag{49}$$

$$\widehat{p}_t^{\mathrm{N}} = (1-\beta(1-q))\left[\widehat{\lambda}_t + \frac{\widehat{\widetilde{p}}_t}{\theta-1} + \widehat{s}_t + \widehat{y}_t\right] + \beta(1-q)E_t\widehat{p}_{t+1}^{\mathrm{N}} \tag{50}$$

$$\widehat{p}_t^{\rm D} = (1-\beta(1-q))\left[\widehat{\lambda}_t + \frac{\widehat{\widetilde{p}}_t}{\theta-1} + \widehat{y}_t\right] + \beta(1-q)\left[E_t\widehat{p}_{t+1}^{\rm D} - E_t\widehat{\pi}_{t+1}\right] \quad (51)$$

$$0 = q\widehat{\widetilde{p}}_t - (1-q)\widehat{\pi}_t \quad (52)$$

$$\widehat{\overline{p}}_t = q\widehat{\widetilde{p}}_t + (1-q)(\widehat{\overline{p}}_{t-1} - \widehat{\pi}_t) \quad (53)$$

Note that from equation (52), we have

$$\widehat{\widetilde{p}}_t = \frac{1-q}{q}\widehat{\pi}_t$$

Plugging this result in (53), we end up with $\widehat{\overline{p}}_t = \widehat{\overline{p}}_{t-1}$, implying that if the economy is started from its steady state level, $\widehat{\overline{p}}_t = 0$ for all t, which we will consider hereafter. This implies that (44) becomes

$$\widehat{y}_t = \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{h}_t$$

From, (50) and (51), we have

$$\widehat{p}_t^{\rm N} - \widehat{p}_t^{\rm D} = (1-\beta(1-q))\widehat{s}_t + \beta(1-q)\left[E_t\widehat{p}_{t+1}^{\rm N} - E_t\widehat{p}_{t+1}^{\rm D} + E_t\widehat{\pi}_{t+1}\right]$$

Further, from (49) and (52), we get that

$$\widehat{\widetilde{p}}_t = \widehat{p}_t^{\rm N} - \widehat{p}_t^{\rm D} = \frac{1-q}{q}\widehat{\pi}_t$$

which then implies that

$$\widehat{\pi}_t = \frac{q(1-\beta(1-q))}{1-q}\widehat{s}_t + \beta E_t\widehat{\pi}_{t+1}$$

Hence we may reduce the system to

$$\zeta_{cc}\widehat{c}_t + \zeta_{cm}\widehat{m}_t - \zeta_{c\ell}\frac{h^\star}{1-h^\star}\widehat{h}_t \quad (54)$$

$$(\zeta_{\ell c} - \zeta_{cc})\widehat{c}_t + (\zeta_{\ell m} - \zeta_{cm})\widehat{m}_t + \left(\alpha - \frac{h^\star}{1-h^\star}(\zeta_{\ell\ell} - \zeta_{c\ell})\right)\widehat{h}_t = \widehat{s}_t + \widehat{a}_t + \alpha\widehat{k}_t \quad (55)$$

$$(\zeta_{mc} - \zeta_{cc})\widehat{c}_t + (\zeta_{mm} - \zeta_{cm})\widehat{m}_t - (\zeta_{m\ell} - \zeta_{c\ell})\frac{h^\star}{1-h^\star}\widehat{h}_t = \frac{\widehat{R}_t}{R^\star - 1} \quad (56)$$

$$\widehat{y}_t = \frac{c^\star}{y^\star}\widehat{c}_t + \frac{x^\star}{y^\star}\widehat{x}_t + \frac{g^\star}{y^\star}\widehat{g}_t \quad (57)$$

$$\widehat{y}_t = \widehat{a}_t + \alpha\widehat{k}_t + (1-\alpha)\widehat{h}_t \quad (58)$$

$$\widehat{z}_t = \widehat{s}_t + \widehat{a}_t + (1-\alpha)\widehat{h}_t - (1-\alpha)\widehat{k}_t \quad (59)$$

$$\widehat{k}_{t+1} = \delta\widehat{x}_t + (1-\delta)\widehat{k}_t \quad (60)$$

$$\widehat{m}_t = \widehat{\mu}_t + \widehat{m}_{t-1} - \widehat{\pi}_t \tag{61}$$

$$\widehat{\lambda}_t = E_t \widehat{\lambda}_{t+1} + (1 - \beta(1 - \delta)) E_t \widehat{z}_{t+1} \tag{62}$$

$$(1 - \beta(1 - \delta)) E_t \widehat{z}_{t+1} = \widehat{R}_t - E_t \widehat{\pi}_{t+1} \tag{63}$$

$$\widehat{\pi}_t = \frac{q(1 - \beta(1 - q))}{1 - q} \widehat{s}_t + \beta E_t \widehat{\pi}_{t+1} \tag{64}$$

Plus some stochastic processes for the forcing variables $\widehat{a}_t$, $\widehat{g}_t$, $\widehat{\mu}_t$.