LECTURE II:
The Stochastic Neoclassical Growth Model in Discrete Time

A. Model

We consider again the Neoclassical Growth Model, introduced in Lecture I. It will be convenient to work with the social planner's version of this model. Recall that in this case the problem is:

$$\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \, u[(1-\delta)k_t + Af(k_t) - (1+g_n)(1+g_z)k_{t+1}]$$

subject to:

$$0 \leq k_{t+1} \leq \frac{(1-\delta)k_t + Af(k_t)}{(1+g_n)(1+g_z)}; \; k_0 \text{ given.}$$

But, in this lecture, we will substitute the assumption that total factor productivity is given by a constant parameter, A, with the assumption that, in any given period, t, total factor productivity is characterized by an exogenous random variable, θ. We assume that θ_t is observed by economic agents at the beginning of period t. Therefore, based on θ_t and since capital (per efficient household) at the beginning of period t, k_t, is given, output in period t:

$$y_t = \theta_t f(k_t) \tag{II.1}$$

is also known. Then, economic agents choose capital at the beginning of next period, k_{t+1}. This observation has two important implications. That is, there are two important differences from the deterministic case. First, the objective function of the representative household can no longer be the same with the deterministic case, because, now, lifetime utility is a random variable based on the future values of the random variable θ_t, (i.e,

$\theta_1, \theta_2, \ldots$). Second, the plan in the deterministic problem is a sequence of numbers. But, now, capital at the beginning of every period of every future period, $t+1$, in the future depends on the sequence of values of the random variable, θ_t, from the present period up to period t. For that matter, we assume that the objective function is the expectation of the lifetime utility of the representative household, conditioned on the current value of the total factor productivity, θ_0, where expectations are taken with respect to the probability distribution of all future values of θ, given θ_0. And, we seek a plan of future capital stocks, such that capital at the beginning of every future period $t+1$ is a function of the past realizations of random variable θ up to period t. That is, the problem we consider is the following:

$$\max_{\{k_{t+1}(\theta^t)\}_{t=0}^{\infty}} E\left(\sum_{t=0}^{\infty} \beta^t u\{(1-\delta)k_t(\theta^{t-1}) + \theta_t f[k_t(\theta^{t-1})] - (1+g_n)(1+g_z)k_{t+1}(\theta^t)\} \middle| \theta_0 \right) \tag{II.2}$$

subject to:

$$0 \le k_{t+1}(\theta^t) \le \frac{(1-\delta)k_t(\theta^{t-1}) + \theta_t f[k_t(\theta^{t-1})]}{(1+g_n)(1+g_z)} \tag{II.3}$$

$$(k_0,\ \theta_0) \text{ given} \tag{II.4}$$

where: $E(\bullet)$ denotes the mathematical expectations operator and $E(\bullet|\theta_0)$ denotes that expectations are with respect to the probability distribution of $\{\theta_{t+1}\}_{t=0}^{\infty}$ conditioned on θ_0; $\theta^t \equiv \{\theta_0, \ldots, \theta_{t-1}, \theta_t\}$ is called the history generated by the random variable θ up to the beginning of period t and characterizes the information available to economic agents at the time k_{t+1} must be chosen. Note, that k_1 is a number and k_{t+1} is a sequence of functions of the information that will be available at the beginning of period, θ^t. For that matter, $\{k_{t+1}(\theta^t)\}_{t=0}^{\infty}$ is called a contingency plan, as the information on which decisions will be made is not as yet determined.

Before introducing the probability distribution of θ^t, it will be convenient to introduce the stochastic equivalent of the deterministic Basic Problem, we considered in Lecture I.

B. The Basic Dynamic Stochastic Problem

Note that the preceding problem is a problem of the form:

$$\max_{\{x_{t+1}(\theta^t)\}_{t=0}^{\infty}} E\{\sum_{t=0}^{\infty} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \,|\, \theta_0\} \qquad \text{(II.5)}$$

subject to:

$$x_{t+1}(\theta^t) \in \Gamma[x_t(\theta^{t-1}), \theta_t] \quad \forall\ t \geq 0 \qquad \text{(II.6)}$$

$$(x_0, \theta_0) \in (X\ x\ \Theta) \text{ given} \qquad \text{(II.7)}$$

where:

x_t endogenous state variable

X set with the possible values of the endogenous state variable

θ exogenous state variable

Θ set of possible values of the exogenous state variable

(x_t, θ_t) describes the state of the system the beginning of period t

Γ $\Gamma : X\ x\ X\ x\ \Theta \rightarrow X$ correspondence that describes the constraints in the transition of the endogenous state variable from one time period to the next (i.e., $\forall (x,\theta) \in X\ x\ \Theta$, $\Gamma(x,\theta)$ is the set of all feasible values of the endogenous state variable at the beginning of the next period, if the value of the state variable at the beginning of the current period is (x,θ))

$\beta \in (0,1)$ time discount factor

$A = \{(x, y, \theta) \in X \ x \ X \ x \, \Theta : y \in \Gamma(x, \theta)\}$ set of all possible combinations of the state of the system and next period's endogenous state variable

$\Phi : A \to \mathbb{R}$ temporal objective function

We shall refer to (II.5) – (II.7) as the Basic Dynamic Stochastic Problem or simply as the Basic Stochastic Problem. Clearly then, the social planner's problem is an example of the Basic Stochastic Problem, where: $x_t = k_t(\theta^{t-1})$, $\mathrm{X} = \mathbb{R}_+$, $\Theta = \mathbb{R}_+$,

$$\Gamma(x,\theta) = \{y \in \mathbb{R}_+ : 0 \le y \le \frac{(1-\delta)x + \theta f(x)}{(1+g_n)(1+g_z)}\}$$

and

$$\Phi(x, y, \theta) = u[(1 - \delta)x + \theta f(x) - (1 + g_n)(1 + g_z)y]$$

C. Probabilities

In general, the specification of probabilities in this context, requires measurability issues that are beyond the scope of this course. However, we can consider some special cases that are indicative both of the problems involved as well as how these problems are resolved in the more general cases.

(i) Finite state space and iid random variables

Suppose that random variable θ_t's can take a limited number of constant values, $\Theta = \{\vartheta_1, \vartheta_2, ..., \vartheta_N\}$ and that the θ_t's are independent to each other and identically distributed (iid) random variables:

$$\Pr(\theta_t = \vartheta_j) = \pi_j \ni 0 \le \pi_j \le 1 \& \sum_{j=1}^{N} \pi_j = 1 \qquad \text{(II.8)}$$

In this case, $\theta^t \in \Theta^t = \mathop{\mathrm{X}}_{i=1}^{t}\Theta$ and for any $\theta^t = (\vartheta_{\alpha_1}, \vartheta_{\alpha_2}, ..., \vartheta_{\alpha_t})$,

$\pi_{\theta_t} \equiv \Pr(\theta^t) = \Pr(\vartheta_{\alpha_1}, \vartheta_{\alpha_2}, ..., \vartheta_{\alpha_t})] = \prod_{i=1}^{t}\pi_{\alpha_i}$. The objective function of the Basic Stochastic Problem is:

$$E\{\sum_{t=0}^{\infty}\beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \,|\theta_0\} \; E\{\sum_{t=0}^{\infty}\beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\} =$$

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^t} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\pi_{\theta^t} =$$

$$\Phi[x_0, x_1, \theta_0] + \beta \sum_{\theta^1 \in \Theta^1} \Phi[x_1, x_2(\theta^1), \theta_1]\pi_{\theta^1} + \ldots + \beta^t \sum_{\theta^t \in \Theta^t} \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\pi_{\theta^t} +$$

$$\beta^{t+1} \sum_{\theta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}]\pi_{\theta^{t+1}} + ...$$

It follows as in Remark of Lecture 1, that if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to:

$$\max_{x_{t+1}(\theta^t)}\{\sum_{\theta^t \in \Theta^t} \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\pi_{\theta^t} + \beta \sum_{\theta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}]\pi_{\theta^{t+1}}\}$$

Since,

$$\sum_{\theta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}]\pi_{\theta^{t+1}} = \sum_{\theta^t \in \Theta^t} \pi_{\theta^t} \sum_{\theta_{t+1} \in \Theta} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^t, \theta_{t+1}), \theta_{t+1}]\pi_{\theta_{t+1}}$$

it follows that if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to:

$$\max_{x_{t+1}(\theta^t)}\left(\sum_{\theta^t \in \Theta^t} \pi_{\theta^t}\{\Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + \beta \sum_{\theta_{t+1} \in \Theta} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^t, \theta_{t+1}), \theta_{t+1}]\pi_{\theta_{t+1}}\}\right)$$

Hence, if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to the preceding problem for the particular value of θ^t that will be known in the beginning of period t,

$$\max_{x_{t+1}}\{\Phi(x_t, x_{t+1}, \theta_t) + \beta \sum_{\theta_{t+1}\in\Theta} \Phi[x_{t+1}, x_{t+2}(\theta_{t+1}), \theta_{t+1}]\pi_{\theta_{t+1}}\} \text{ or}$$

$$\max_{x_{t+1}}\{\Phi(x_t, x_{t+1}, \theta_t) + \beta\ E\ \Phi[x_{t+1}, x_{t+2}(\theta_{t+1}), \theta_{t+1}]\} \qquad \text{(II.9)}$$

(ii) <u>Infinite state space and iid random variables</u>

Suppose that random variable θ_t's take values in the interval of real numbers $\Theta = [\underline{\vartheta}, \overline{\vartheta}]$, so that:

$$\Pr(\theta_t \le \tilde{\vartheta}) = \Psi(\tilde{\vartheta}) = \int_{\underline{\vartheta}}^{\tilde{\vartheta}} \psi(\vartheta)d\vartheta \qquad \text{(II.10)}$$

where $\psi(\vartheta)$ is a strictly positive real valued function, such that $\Psi(\vartheta)$ is continuous and strictly increasing in Θ, $\Psi(\vartheta) \in [0,1]$, $\Psi(\vartheta) \to 0$ as $\vartheta \to \underline{\vartheta}$ and $\Psi(\vartheta) \to 1$ as $\vartheta \to \overline{\vartheta}$. (Throughout these notes integrals are assumed to exist.) Recall that $\psi(\vartheta)$ and $\Psi(\vartheta)$ are called probability density and cumulative probability distribution function, respectively. Recall that for a continuous random variable, like θ_t, its expected value is defined as:

$$E(\theta_t) = \int_{\underline{\vartheta}}^{\overline{\vartheta}} \vartheta\psi(\vartheta)d\vartheta = \int_{\underline{\vartheta}}^{\overline{\vartheta}} \vartheta d\Psi(\vartheta)$$

where the first and second integrals above are the simple Riemann and the Riemann-Stieltjes integral, respectively (See, e.g., Bartle (1976), Section 29). Throughout these notes these integrals are assumed to exist (i.e., the integrating functions are integrable in the above sense.. Also, recall that for any collection of continuous random variables like ϑ^t, the joint distribution function is defined by:

$\tilde{\Psi}(\vartheta^t) = \Pr(\theta^t \le \vartheta^t) = \Pr(\theta_1 \le \vartheta_{\alpha_1}, \theta_2 \le \vartheta_{\alpha_2}, ..., \theta_t \le \vartheta_{\alpha_t})$, where: $\theta^t = (\theta_1, \theta_2, ..., \theta_t)$, $\theta^t \in \Theta^t = \mathop{\mathrm{X}}_{i=1}^{t} \Theta$ and $\vartheta^t = (\vartheta_{\alpha_1}, \vartheta_{\alpha_2}, ..., \vartheta_{\alpha_t})$. The components of θ^t are independently distributed. This means that:

$$\begin{aligned}
&\tilde{\Psi}(\vartheta^t) = \Pr(\theta^t \le \vartheta^t) = \Pr(\theta_1 \le \vartheta_{\alpha_1}, \theta_2 \le \vartheta_{\alpha_2}, ..., \theta_t \le \vartheta_{\alpha_t}) = \\
&\Pr(\theta_1 \le \vartheta_{\alpha_1})\Pr(\theta_2 \le \vartheta_{\alpha_2})...\Pr(\theta_t \le \vartheta_{\alpha_t}) = \Psi(\vartheta_{\alpha_1})\Psi(\vartheta_{\alpha_2})...\Psi(\vartheta_{\alpha_t}) = \\
&\int_{\underline{\vartheta}}^{\vartheta_{\alpha_1}} \psi(\vartheta)d\vartheta \int_{\underline{\vartheta}}^{\vartheta_{\alpha_2}} \psi(\vartheta)d\vartheta ... \int_{\underline{\vartheta}}^{\vartheta_{\alpha_t}} \psi(\vartheta)d\vartheta
\end{aligned} \qquad \text{(II.11)}$$

Now, as in Case (i), the objective function of the Basic Stochastic Problem is:

$$E\{\sum_{t=0}^{\infty} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \big| \theta_0\} \; E\{\sum_{t=0}^{\infty} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\} =$$

$$\sum_{t=0}^{\infty} \int_{\vartheta^t \in \Theta^t} \beta^t \Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t), \vartheta_{\alpha_t}] \Pr(\theta^t \le \vartheta^t) d\vartheta^t =$$

$$\Phi[x_0, x_1, \theta_0] + \beta \int_{\underline{\vartheta}}^{\bar{\vartheta}} \Phi[x_1, x_2(\theta_0, \vartheta_{\alpha_1}), \vartheta_{\alpha_1}] \psi(\vartheta_{\alpha_1}) d\vartheta_{\alpha_1} + \ldots +$$

$$\beta^t \int_{\vartheta^t \in \Theta^t} \beta^t \Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t), \vartheta_{\alpha_t}] \Pr(\theta^t \le \vartheta^t) d\vartheta^t$$

$$+\beta^{t+1} \int_{\vartheta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\vartheta^t), x_{t+2}(\vartheta^{t+1}), \vartheta_{\alpha_{t+1}}] \Pr(\theta^{t+1} \le \vartheta^{t+1}) d\vartheta^{t+1} + ...$$

It follows as in Remark of Lecture 1, that if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to:

$$\begin{aligned}
&\max_{x_{t+1}(\theta^t)} \{ \int_{\vartheta^t \in \Theta^t} \Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t), \vartheta_{\alpha_t}] \Pr(\theta^t \le \vartheta^t) d\vartheta^t + \\
&\beta \int_{\vartheta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\vartheta^t), x_{t+2}(\vartheta^{t+1}), \vartheta_{\alpha_{t+1}}] \Pr(\theta^{t+1} \le \vartheta^{t+1}) d\vartheta^{t+1} \}
\end{aligned}$$

Since,

$$\int_{\vartheta^{t+1}\in\Theta^{t+1}} \Phi[x_{t+1}(\vartheta^t), x_{t+2}(\vartheta^{t+1}), \vartheta_{\alpha_{t+1}}]\Pr(\theta^{t+1} \leq \vartheta^{t+1})d\vartheta^{t+1} =$$

$$\int_{\vartheta^t\in\Theta^t} \left\{ \int_{\underline{\vartheta}}^{\bar{\vartheta}} \Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t, \vartheta_{\alpha_{t+1}}), \vartheta_{\alpha_{t+1}}]\Pr(\theta_{t+1} \leq \vartheta_{t+1})d\vartheta_{\alpha_{t+1}} \right\} \Pr(\theta^t \leq \vartheta^t)d\vartheta^t$$

it follows that if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to:

$$\max_{x_{t+1}(\theta^t)}\{ \int_{\vartheta^t\in\Theta^t} \left(\Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t), \vartheta_{\alpha_t}] + \beta \left\{ \int_{\underline{\vartheta}}^{\bar{\vartheta}} \Phi[x_t(\vartheta^{t-1}), x_{t+1}(\vartheta^t, \vartheta_{\alpha_{t+1}}), \vartheta_{\alpha_{t+1}}]\Pr(\theta_{t+1} \leq \vartheta_{t+1})d\vartheta_{\alpha_{t+1}} \right\} \right) \Pr(\theta^t \leq \vartheta^t)d\vartheta^t \}$$

Hence, if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to the preceding problem for the particular value of θ^t that will be known in the beginning of period t,

$$\max_{x_{t+1}}\{\Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + \beta\int_{\underline{\vartheta}}^{\bar{\vartheta}} \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t, \vartheta_{\alpha_{t+1}}), \vartheta_{\alpha_{t+1}}]\Pr(\theta_{t+1} \leq \vartheta_{t+1})d\vartheta_{\alpha_{t+1}}$$

or

$$\max_{x_{t+1}}\{\Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] + \beta\int_{\underline{\vartheta}}^{\bar{\vartheta}} \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t, \vartheta), \vartheta]\psi(\vartheta)d\vartheta\} \qquad \text{(II.12)}$$

or

$$\max_{x_{t+1}}\{\Phi(x_t, x_{t+1}, \theta_t) + \beta\ E\ \Phi[x_{t+1}, x_{t+2}(\theta_{t+1}), \theta_{t+1}]\}$$

(iii) <u>Finite state space and Markov chains</u>

Suppose that random variable θ_t's take their values on $\Theta = \{\vartheta_1, \vartheta_2, ..., \vartheta_N\}$ and that for any given value of θ_t, say θ_j, the conditional probability of θ_{t+1} is given by:

$$\Pr(\theta_{t+1} = \vartheta_i \left| \theta_t = \vartheta_j \right.) = \pi_{ij} \ni 0 \le \pi_{ij} \le 1 \& \sum_{j=1}^{N} \pi_{ij} = 1 \tag{II.13}$$

Recall from basic probability theory that for any random variable, ω, that takes its values on the finite set Ω, and any two events A and B (i.e., subsets of Ω), the conditional probability of A, given B, is given by:

$$\Pr(\mathrm{A}\left|\mathrm{B}\right.) = \frac{\Pr(\mathrm{A} \cap \mathrm{B})}{\Pr(\mathrm{B})} \quad \text{, provided that } \Pr(\mathrm{B}) \neq 0 \tag{II.14}$$

Applying this property of conditional probability on the sequence of random variables θ^t, we have:

$$\begin{aligned} &\Pr(\theta^{t+1}\left|\theta_0\right.) = \Pr(\theta^t \cap \theta_{t+1}) = \Pr(\theta_{t+1}\left|\theta^t\right.)\Pr(\theta^t) = \Pr(\theta_{t+1}\left|\theta_t\right.)\Pr(\theta^t) \\ &= ... = \Pr(\theta_{t+1}\left|\theta_t\right.)\Pr(\theta_t\left|\theta_{t-1}\right.)...\Pr(\theta_1\left|\theta_0\right.) = \prod_{\nu=0}^{t} \pi_{\alpha_{\nu+1}\alpha_\nu} \equiv \pi_{\theta^{t+1}} \end{aligned} \tag{II.15}$$

where, we have used the fact that $\Pr(\theta_0) = 1$. In this case, the objective function of the Basic Stochastic Problem, is given by:

$$E\{\sum_{t=0}^{\infty} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t]\} =$$

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^t} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \Pr(\theta^t \left|\theta_0\right.)$$

$$\sum_{t=0}^{\infty} \sum_{\theta^t \in \Theta^t} \beta^t \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \pi_{\theta^t} =$$

$$\Phi[x_0, x_1, \theta_0] + \beta \sum_{\theta^1 \in \Theta^1} \Phi[x_1, x_2(\theta^1), \theta_1] \pi_{\theta^1} + \ldots + \beta^t \sum_{\theta^t \in \Theta^t} \Phi[x_t(\theta^{t-1}), x_{t+1}(\theta^t), \theta_t] \pi_{\theta^t}$$

$$+ \beta^{t+1} \sum_{\theta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}] \pi_{\theta^{t+1}} + ...$$

Since,

$$\sum_{\theta^{t+1} \in \Theta^{t+1}} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}] \pi_{\theta^{t+1}} = \sum_{\theta^t \in \Theta^t} \pi_{\theta^t} \sum_{\theta_{t+1} \in \Theta} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^t, \theta_{t+1}], \theta_{t+1}] \Pr(\theta_{t+1}\left|\theta_t\right.)$$

it follows as in Case (i) that if if $x_{t+1}(\theta^t)$ is part of the solution to the Basic Stochastic Problem, it must be a solution to:

$$\max_{x_{t+1}}\{\Phi(x_t, x_{t+1}, \theta_t) + \beta \sum_{\theta_{t=1}\in\Theta} \Phi[x_{t+1}(\theta^t), x_{t+2}(\theta^{t+1}), \theta_{t+1}]\Pr(\theta_{t+1}|\theta_t)\}$$

or

$$\max_{x_{t+1}}\{\Phi(x_t, x_{t+1}, \theta_t) + \beta\ E\ \Phi[x_{t+1}, x_{t+2}(\theta_{t+1}), \theta_{t+1}]|\theta_t\} \qquad \text{(II.16)}$$

(iv) Infinite state space and Markov chains:

Remark 1:

In Cases (i) – (iv), if there exists a solution to the Basic Stochastic Problem and contingency plan $\left\{x^*_{t+1}(\theta^t)\right\}_{t=0}^{\infty}$ is such a solution. Then, $\forall t \geq 0$, $x^*_{t+1}(\theta_t)$ must be a solution to the following problem:

$$\max_{x_{t+1}}\{\Phi(x^*_t(\theta_{t-1}), x_{t+1}, \theta_t) + \beta\ E\ \Phi[x_{t+1}, x^*_{t+2}(\theta_{t+1}), \theta_{t+1}]|\theta_t\} \qquad \text{(II.17)}$$

subject to:

$$x_{t+1} \in \Gamma[x^*_t(\theta_{t-1}), \theta_t] \qquad \text{(II.18)}$$

$$x^*_{t+2}(\theta_{t+1}) \in \Gamma[x_{t+1}, \theta_{t+1}] \qquad \text{(II.19)}$$

Then, the following follows by the linearity of conditional expectations.

Theorem 1 (Necessary Conditions):

If $\Phi: A \to \Re$ is continuously differentiable in the interior of A and $\left\{x^*_{t+1}(\theta^t)\right\}_{t=0}^{\infty}$ is an interior solution to the Basic Stochastic Problem, (II.5)-(II.7)(i.e., $x^*_{t+1}(\theta^t) \in \text{int}\left[\Gamma\left(x^*_t(\theta^{t-1}, \theta_t\right)\right] \quad \forall t \geq 0$), then the contingency plan $\left\{x^*_{t+1}(\theta^t)\right\}_{t=0}^{\infty}$ must satisfy the following:

$$\Phi_2\left(x^*_t(\theta^{t-1}), x^*_{t+1}(\theta^t), \theta_t\right) + \beta\ E\{\Phi_1\left(x^*_{t+1}(\theta^t), x^*_{t+2}(\theta^{t+1}), \theta_{t+1}\right)|\theta_t\} = 0 \qquad \forall t \geq 0 \qquad \text{(II.20)}$$

Condition (I.27) is the, so called, Stochastic Euler Condition.

Theorem 2 (Sufficient Conditions):

Suppose that $\Phi : A \to \Re$ *is continuously differentiable in the interior of A and concave and* $\left\{x_{t+1}^{*}(\theta^{t})\right\}_{t=0}^{\infty}$ *is an interior solution to the Basic Stochastic Problem, (II.5)-(II.7)(i.e.,* $x_{t+1}^{*}(\theta^{t}) \in \operatorname{int}\left[\Gamma\left(x_{t}^{*}(\theta^{t-1}, \theta_{t}\right)\right] \quad \forall t \geq 0$*), then if the contingency plan* $\left\{x_{t+1}^{*}(\theta^{t})\right\}_{t=0}^{\infty}$ *satisfies the Euler Condition (II.18), the Initial condition, as well as the following:*

$$\lim_{\mathrm{T}\to\infty} \beta^{\mathrm{T}} E\{\Phi_1\left(x_{\mathrm{T}}^{*}(\theta^{T-1}), x_{\mathrm{T}}^{*}(\theta^{T}), \theta_{T}\right)\left(x_{\mathrm{T}}(\theta^{T}) - x_{\mathrm{T}}^{*}(\theta^{T})\right) \geq 0\,, \tag{II.21}$$

for any feasible contingency plan, is a solution to the Basic Stochastic Problem.

Proof: Left as an exercise for the students

Remark 2:

Condition (II.19) is the stochastic Transversality condition. If $\Phi_1\left(x_{\mathrm{T}}(\theta^{t-1}), x_{\mathrm{T}+1}(\theta^{t})\right)$, $x_{\mathrm{T}+1}(\theta^{t}) \geq 0$, (II.19) can be replaced by the easily verifiable condition:

$$\lim_{\mathrm{T}\to\infty} \beta^{\mathrm{T}} E\{\Phi_1\left(x_{\mathrm{T}}^{*}(\theta^{t-1}), x_{\mathrm{T}+1}^{*}(\theta^{t})\right)\left(x_{\mathrm{T}}^{*}(\theta^{t})\right) = 0 \tag{II.22}$$

Remark 3: The Euler and Transversality conditions associated with the Stochastic Neoclassical Growth Model are:

$$u_c\left(c\ (\theta_t)\right)[-(1+g_n)(1+g_z)] + \beta\ E\{u_c\left(c(\theta_{t+1})\right)[(1-\delta) + a\theta_{t+1}k(\theta_{t+1})]\left|\theta_t\right\} = 0\,; \ \forall t \in \mathbb{N}_+ \tag{II.23}$$

$$\beta^{T} E\left[u_c\left(c_T(\theta_T)\right)k_{T+1}(\theta_T)\left|\theta_0\right.\right] \to 0 \text{ as } T \to \infty \tag{IV.23}$$

Note, that given the strict concavity of the temporal utility function and the non-negativity of consumption, these conditions are sufficient.

Exercise 2: Suppose that $x_t \in \mathbb{R}_+^n, \theta_t \in \mathbb{R}_+^m$, and :

$$\Phi(x, y, \theta) = (1', \theta', x', y')\,\mathrm{A}(1, \theta, x, y)'$$

where ´denotes transposition of a vector or matrix. Then, the Euler condition has the form:

$$B_2 E_t(x_{t+2}) + B_1 E_t(x_{t+1}) + B_0 E_t(x_t) = C + D_1 E_t(\theta_{t+1}) + D_0 E_t(\theta_t)$$

for appropriately defined matrices, $B_2, B_1, B_0, C, D_1, D_0$.

Exercise 3: (a) Show that the solution to the Stochastic Neoclassical Growth Model with:

$u(c) = lnc;\ f(k) = k^{\alpha},\ \alpha \in (0,1);\ \delta = 1\ and\ i.i.d.\ \theta^t$ is given by:

$$k_{t+1} = \alpha\beta\theta_t k_t^{\alpha}$$

(b) Suppose that $\ln\theta_t \sim N(\mu, \sigma^2)$. Find the probability distribution of the optimal contingency plan $\{k_{t+1}\}_{t=0}^{\infty}$ and show that:

$$k_{t+1} \sim N(\mu_t, \sigma_t^2)$$

for appropriately defined means μ_t and variances σ_t^2.

(c) Calculate the limiting values of these means and variances as $t \to \infty$ .variables