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# Introduction to some Topological Notions

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Basic Calculus has shaped our intuition on what the distance between two real numbers is, and more generally between elements of $\mathbb{R}^N$ with $N \in \mathbb{N}^*$, in terms of the corresponding $N$-norms. Let $x, y \in \mathbb{R}$, then we instinctively think of their distance simply as their absolute difference, $|x - y|$. Let $x, y \in \mathbb{R}^2$, then their distance is given by applying the Pythagorean Theorem, $\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. And so on...

In Real Analysis, we study metric spaces and see that we can generalize the idea of the distance between elements of $\mathbb{R}$ or $\mathbb{R}^N$. We realize that the above mentioned absolute distance, the Euclidean distance, etc., are just particular *metric functions* that we can endow these sets with. We realize that we can use other appropriate functions to define the notion of distance between their elements, as long as they maintain a number of desirable properties on them (i.e., positivity, separateness, symmetry, and the triangle inequality).

We also realize that we can consider other kinds of *structured sets*, characterized by other, more "exotic", *carrier sets* and the appropriate *metric functions*, which we call *metric spaces.*

General Topology abstracts away from the idea of a numerically quantifiable distance and replaces *metric functions* with the so called *topologies.* A *topology* can be paired up with a *carrier set* to form what is called a called *topological space.*

Here, we will first be introduced to $d$-openness and $d$-closedness in metric spaces. Next, we will define topologies, $\tau$, which are sets of subsets of the carrier set. A subset of the carrier set is considered to be open with respect to $\tau$, by merit of being an element of $\tau$ (not to be confused with $d$-openness with respect to some metric, $d$). We will then return to metric spaces and discuss sequential convergences and continuity, and finally prove a lemma that allows us to equivalently characterize the continuity of a function in terms of both metric spaces and topological spaces of its domain and co-domain.

*Please report any typos, mistakes, or even suggestions at `zaverdasd@aueb.gr`.

# Openness (and closedness) in metric spaces

**Definition**

Let $(X, d)$ be a metric space and $A$ a subset of $X$ (not necessarily non-empty). $A$ is a $d$-open subset of $X$ (i.e. open with respect to the metric $d$) iff

$$\forall x \in A, \exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

The $x$ subscript on $\varepsilon_x$ means that generally this radius may depend on the element in question.

For any metric space, $(X, d)$, there is a number of examples of $d$-open subsets of the carrier set, $X$. First, there is the carrier set, $X$, itself, because for every one of its elements we can find a radius such that the constructed $d$-open ball is a subset of the carrier set (and in fact any radius will do). For now, we will axiomatically say that the empty set, $\varnothing$, is also a $d$-open subset of the carrier set of $X$, but we will also show why that is when we will talk about sequential $d$-convergence as a way of showing $d$-closedness. Finally, it can be shown that $d$-open balls are also $d$-open subsets of their carrier sets.

**Lemma**

Let $(X, d)$ be a metric space. Any $d$-open ball in it is a $d$-open subset of $X$.

**Proof**

Choose an arbitrary $x \in X$ and radius $\varepsilon > 0$ and define the $d$-open ball $\mathcal{O}_d(x, \varepsilon)$.
For some $y \in \mathcal{O}_d(x, \varepsilon)$ define $\delta \coloneqq \varepsilon - d(x, y) > 0$.
Then $\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$ (to see why see Problem Set 2 Exercise 3).
So $\mathcal{O}_d(x, \varepsilon)$ is a $d$-open subset of $X$. Since $x$ and $\varepsilon$ were chosen arbitrarily, every $d$-open ball is a $d$-open of subset of $X$. ■

Furthermore, the following properties hold with respect to $d$-openness:

- Arbitrary unions of $d$-open sets are $d$-open.

  It is easy to intuitively understand why. If for an element of a $d$-open set, $x \in A$, we can find a radius, $\varepsilon_x > 0$, such that $\mathcal{O}_d(x, \varepsilon_x)$ lies entirely in $A$, then this would hold for any arbitrary union of $A$ with other sets. This holds for any $x \in A$ and if all sets in the union are $d$-open, then the union is $d$-open.

- Finite intersections of $d$-open sets are $d$-open.

To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric, $d_u$ (absolute difference).

Let $A_n$ be subsets of $\mathbb{R}$ such that

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \forall\, n \in \mathbb{N}^*$$

With respect to the usual metric on $\mathbb{R}$, all $A_n$ are $d_u$-open subsets of $\mathbb{R}$.

Now notice that there is an infinite number of such sets and their only common element is 0. So their intersection (an infinite intersection of $d_u$-open sets) is $\{0\}$.

But also notice that $\forall \varepsilon > 0\ \mathcal{O}_{d_u}(0, \varepsilon) \supset \{0\}$. So $\nexists\, \varepsilon > 0 : \mathcal{O}_{d_u}(0, \varepsilon) \subseteq \{0\}$.

Thus we have found an infinite intersection of $d_u$-open sets that is not $d_u$-open.

**Definition**

Let $(X, d)$ be a metric space. Any subset, $A$, of $X$ is termed *a $d$-closed subset of $X$* if its complement, $A'$ (i.e. $A' = X \setminus A$), is a $d$-open subset of $X$.

This definition has a few implications. First, the carrier set, $X$, is a $d$-closed subset of itself, since the empty set is a $d$-open subset of $X$. Secondly, the empty set, $\varnothing$, is a $d$-closed subset of $X$, since $X$ is a $d$-open subset of itself. Finally, it can be shown that $d$-closed balls are also $d$-closed subsets of $X$.

**Lemma**

Let $(X, d)$ be a metric space. Any $d$-closed ball in it is a $d$-closed subset of $X$.

**Proof**

Consider an arbitrary $d$-closed ball in $X$, $\mathcal{O}_d[x, \varepsilon]$. It suffices to show that its complement, $\mathcal{O}'_d[x, \varepsilon]$, is a $d$-open subset of $X$.

Let $y \in \mathcal{O}'_d[x, \varepsilon] \iff d(x, y) > \varepsilon$. We want to find a radius, $\delta$, such that

$$\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}'_d[x, \varepsilon]$$

$$\mathcal{O}_d(y, \delta) \bigcap \mathcal{O}_d[x, \varepsilon] = \varnothing$$

$$d(x, z) > \varepsilon, \forall\, z \in \mathcal{O}_d(y, \delta)$$

Now, define $\delta := d(x, y) - \varepsilon > 0$ and let $z \in \mathcal{O}_d(y, \delta)$. Naturally, the dual of the triangle inequality holds between the chosen $x$, $y$, and $z$ (see Problem Set 1 Exercise 5)

$$\begin{aligned}
&d(x,z) \geq |d(x,y) - d(y,z)| \\
&d(x,z) \geq |\varepsilon + \delta - d(y,z)| \\
&d(x,z) \geq \varepsilon + |\delta - d(y,z)| \\
&d(x,z) > \varepsilon
\end{aligned}$$

Since $z$ was chosen arbitrarily, the above holds for all $z \in \mathcal{O}_d(y,\delta)$. So there exists a $\delta$ for $y$ such that $d(x,z) > \varepsilon$, $\forall z \in \mathcal{O}_d(y,\delta)$.
Since $y$ was chosen arbitrarily, the above holds for all $y \in \mathcal{O}'_d[x,\varepsilon]$. So $\mathcal{O}'_d[x,\varepsilon]$ is a $d$-open subset of $X$ and $\mathcal{O}_d[x,\varepsilon]$ is $d$-closed.
Since $\mathcal{O}_d[x,\varepsilon]$ was chosen arbitrarily, the above holds for every $d$-closed ball. ■

Furthermore, the following properties hold with respect to $d$-closedness:

- Finite unions of $d$-closed sets are $d$-closed.

  To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric, $d_u$.

  Let $A_n$ be subsets of $\mathbb{R}$ such that

  $$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \forall\, n \in \mathbb{N}^*$$

  All $A_n$ are $d_u$-closed subsets of $\mathbb{R}$.

  Now notice that there is an infinite number of such sets and that

  $$\bigcup_{n=2}^{\infty} A_n = (0,1)$$

  So their union (an infinite union of $d_u$-closed sets) is $B := (0,1)$.

  But also notice that $B' = (-\infty, 0] \cup [1, +\infty)$ and that for its elements 0 and 1 there do not exist a radii $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that $\mathcal{O}_{d_u}(0,\varepsilon_0) \subseteq B'$ and $\mathcal{O}_{d_u}(1,\varepsilon_1) \subseteq B'$, respectively. So $B'$ is not an $d_u$-open subset of $\mathbb{R}$ and its complement $B$, an infinite union of $d_u$-closed sets, is not $d_u$-closed.

- Arbitrary intersections of $d$-closed sets are $d$-closed.

> To intuitively see why, first consider a $d$-closed subset of some carrier set. Then its complement is $d$-open. Now notice that the complement of the intersection of two sets is the union of their complements. The complement of an arbitrary intersection of $d$-closed sets is the union of the arbitrary complements, which are $d$-open. So the union is $d$-open and the intersection is $d$-closed.

It is important to realize that $d$-openness and $d$-closedness are not necessarily mutually exclusive, nor are they opposites. A set can be $d$-open, $d$-closed, both $d$-open and $d$-closed, or neither $d$-open or $d$-closed.

For example, consider the set of real numbers endowed with the usual metric, $d_u$. By our intuition (but it can also be shown rigorously) $(0, 1)$ is a $d_u$-open subset of $\mathbb{R}$ but not $d_u$-closed. $[0, 1]$ is a $d_u$-closed subset of $\mathbb{R}$ but not $d_u$-open. $[0, 1)$ is neither a $d_u$-open nor a $d_u$-closed subset of $\mathbb{R}$. Finally, endow $\mathbb{R}$ with the discrete metric, $d_\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$d_\delta(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

then $\{0\}$ is a $d_\delta$-open subset of $\mathbb{R}$, because $\mathcal{O}_{d_\delta}\left(0, \frac{1}{2}\right) \subseteq \{0\}$ and it is also $d_\delta$-closed because $\{0\}' = \mathbb{R}^*$ is a $d_\delta$-open subset of $\mathbb{R}$ (because $\mathcal{O}_{d_\delta}\left(x, \frac{1}{2}\right) \subseteq \mathbb{R}^*$, $\forall x \in \mathbb{R}^*$).

# Topologies

### Definition

A topology, $\tau$, on a set, $X$, is a collection of subsets of $X$ that satisfy the following properties:

1. $\varnothing$ and $X$ belong to $\tau$ ($\varnothing, X \in \tau$)

2. any *arbitrary* union of elements of $\tau$ is also an element of $\tau$ ($\tau$ is closed with respect to arbitrary unions) ($A_i \in \tau$, $\forall\, i \in \mathcal{I} \Rightarrow \bigcup_{i\in\mathcal{I}} A_i \in \tau$)

3. any *finite* intersection of elements of $\tau$ is also an element of $\tau$ ($\tau$ is closed with respect to finite intersections) ($A_i \in \tau$, $\forall\, i \in \mathcal{I}$ finite $\Rightarrow \bigcap_{i\in\mathcal{I}} A_i \in \tau$)

The pair $(X, \tau)$ is termed *topological space*.

In a topological space, $(X, \tau)$, any subset of $X$, $A \subseteq X$, that also belongs to the chosen topology, $\tau$, ($A \subseteq \tau$) is by definition an open subset of $X$ (and $A'$, its complement in $X$, is closed). A more vague definition of a topology could be that it is *a collection of "open" subsets of* $X$. Notice that we are no longer talking about $d$-openness, because openness is no longer determined by a metric function, but by the chosen topology, $\tau$.

Additionally, a metric, $d$, on $X$ can be also used to generate a topology, $\tau_d$, on $X$.

### Lemma

For every metric function, $d$, that a non-empty set, $X$, is endowable with, there exists an implied topology, $\tau_d$, where

$$\tau_d = \{A \subseteq X : A \text{ is } d\text{-open}\}$$

### Proof

For $\tau_d$ to be a valid topology on $X$, it has to satisfy the properties 1-3 of topologies. The definition of $\tau_d$ basically says that every $d$-open subset of $X$ belongs to it. So

1. $\varnothing$ and $X$ belong to $\tau_d$ because they are $d$-open subsets of $X$.

2. Since all elements of $\tau_d$ are $d$-open subsets of $X$ and arbitrary unions of $d$-open sets are $d$-open, then arbitrary unions of elements of $\tau_d$ also belong to $\tau_d$.

3. Since all elements of $\tau_d$ are $d$-open subsets of $X$ and finite intersections of $d$-open sets are $d$-open, then finite intersections of elements of $\tau_d$ also belong to $\tau_d$.

Thus, $\tau_d$ is a topology on $X$. ■

An example of a topology generated by a metric is the discrete topology, $\tau_\delta$, which is generated by the discrete metric, $d_\delta$.

Let $(X, d_\delta)$ be a metric space and $A$ an arbitrary subset of $X$. Then for any element, $x$, in $A$ there exists a positive radius that is less than one, $0 < \varepsilon < 1$, such that $\mathcal{O}_{d_\delta}(x, \varepsilon) = \{x\} \subseteq A$. So $A$ is $d_\delta$-open and since it was chosen arbitrarily, every subset of $X$ is open with respect to the discrete metric.

So the discrete topology of a set is its powerset, $\tau_\delta = P(X)$.

However, not all topologies can be generated by a metric. One such example can be the indiscrete topology, $\tau_I = \{\varnothing, X\}$, on a non singleton set. Let $X = \{a, b\}$ with $a \neq b$, and assume that there exists a metric $d$ that we can endow $X$ with and that it can produce the indiscrete topology, $\tau_I$. So we want $\exists\, d \mid \tau_I \equiv \{x \in X : x \text{ is } d\text{-open}\}$.

Define $\varepsilon := d(a, b)$ and the $d$-open ball $\mathcal{O}_d(a, \varepsilon)$.

Notice that $\mathcal{O}_d(a, \varepsilon) \in \tau_I$, since it is a $d$-open subset of $X$ and $\tau_I$ is the collection of $d$-open subsets of $X$ (since it is generated by $d$).

Also, $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$ since open balls always contain their center.

Finally, $\mathcal{O}_d(a, \varepsilon) \neq X$ since it does not include $b$.

Thus, $\mathcal{O}_d(a, \varepsilon) \in \tau_I = \{\varnothing, X\}$ and $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$ and $\mathcal{O}_d(a, \varepsilon) \neq X$. Contradiction!

So $\tau_I$ cannot always be generated by a metric. We can, thus, say that *not all topological spaces are metrizable.*

Finally, not all collections of subsets constitute topologies. To see this, let $X = \{a, b, c\}$ and a collection of subsets, $\tau = \{\varnothing, \{a, b\}, \{b, c\}, X\}$. Then the intersection between $\{a, b\}$ and $\{b, c\}$ is $\{b\} \notin \tau$. So $\tau$ violates property 3 and is not a topology on $X$.

We can also dually define a topology, $\tau_d^*$, such that $\tau_d^* = \{A \subseteq X, A \text{ is } d\text{-closed}\}$ and it contains the same informational context as $\tau_d$. Contrary to $\tau_d$, we want $\tau_d^*$ to be closed under *finite* unions and *arbitrary* intersections of its elements.

**Definition**

Let $(X, \tau)$ be a topological space and $x \in X$, then

$$\tau(x) = \{A \in \tau, x \in A\}$$

is called a *neighbouring system of $x$.*

# Convergence and Continuity in Metric Spaces

**Definition**

Let $(X, d)$ be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in $X$. Then we say that $x \in X$ is a $d$-limit of $(x_n)_{n\in\mathbb{N}}$ iff

$$\forall\, \varepsilon > 0,\, \exists\, n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \quad x_n \in \mathcal{O}_d(x, \varepsilon)$$

and can denote $x = d - \lim(x_n)$ and equivalently write $x_n \to x$ ($x_n$ tends to $x$).

**Lemma**

Let $(X, d)$ be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence in $X$, and $x \in X$. Consider $\tau_d$ as the topology on $X$ generated by $d$. Then $x_n \to x$ iff $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $A$.

**Proof**

Suppose that $x$ is the $d$-limit of $(x_n)_{n\in\mathbb{N}}$.

Choose an arbitrary $A \in \tau_d(x) \subseteq \tau_d$. By definition $A$ is a $d$-open subset of $X$ and $x \in A$. So

$$\exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

But because $x_n \to x$

$$\exists\, n_{\varepsilon_x} \in \mathbb{N} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

i.e. almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{O}_d(x, \varepsilon_x)$ and since $\mathcal{O}_d(x, \varepsilon_x) \subseteq A$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $A$. Since $A$ was chosen arbitrarily, this holds for all such $A \subseteq \tau_d(x)$.

For the converse, suppose that $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $A$.

Since $\mathcal{O}_d(x, \varepsilon)$ is a $d$-open subset of $X$ for all $\varepsilon > 0$ and it includes $x$, it follows that $\mathcal{O}_d(x, \varepsilon) \in \tau_d(x)$, which drives the result. ■

**Lemma**

Let $(X, d)$ be a metric space. Every $d$-convergent sequence in $X$ has a unique $d$-limit.

**Proof**

Let $(x_n)_{n\in\mathbb{N}}$ be a $d$-convergent sequence in $X$ with $x = d - \lim(x_n)$, $y = d - \lim(x_n)$, and $x \neq y$.

By separateness we know that $d(x, y) > 0$ so there exist $0 < \varepsilon_x < d(x, y)$ and $0 < \varepsilon_y < d(x, y)$ such that

$$\mathcal{O}_d(x, \varepsilon_x) \bigcap \mathcal{O}_d(y, \varepsilon_y) = \varnothing$$

Since $x_n \to x$ for this particular $\varepsilon_x$

$$\exists\, n_{\varepsilon_x} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

thus only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(x, \varepsilon_x)$.
Analogously, since $x_n \to y$ only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(y, \varepsilon_y)$.
But $\mathcal{O}_d(x, \varepsilon_x)$ and $\mathcal{O}_d(y, \varepsilon_y)$ are disjoint sets. Contradiction! ■

Observe that we need the separateness property to prove the uniqueness of $d$-limits. Thus, $d$-limits are not necessarily unique in pseudometric spaces.

Furthermore, the uniqueness of limits cannot necessarily be generalized in topological spaces that are not generated by a metric. Consider the example of the indiscrete topological space where the carrier set is not a singleton, e.g. $(X, \tau_I)$ with $X = \{a, b\}$, $a \neq b$, and $\tau_I = \{\varnothing, X\}$. Consider any sequence, $(x_n)_{n\in\mathbb{N}}$, in $X$. Here, $a$ is a limit of $(x_n)_{n\in\mathbb{N}}$ because the neighbouring system of $a$ with respect to $\tau_I$ comprises only of $X$ itself

$$\tau_I(a) = \{X\}$$

and the entire sequence is in $X$.

Similarly, $\tau_I(b) = \{X\}$ and $b$ is a limit. So in indescrete spaces limits may not be unique.

**Lemma**

Let $(X, d)$ be a metric space. $A \subseteq X$ is a $d$-closed subset of $X$ iff $\forall (x_n)_{n\in\mathbb{N}}$, with $x_n \in A$, $\forall n \in \mathbb{N}$ and $x_n \to x$ with respect to $d$, then $x \in A$.

**Proof**

Let $A \subseteq X$ be such that for all sequences, $(x_n)_{n\in\mathbb{N}}$, in $A$ with $d - \lim x_n = x$ it holds that $x \in A$. Also assume that $A$ is not $d$-closed. Thus,

$$\begin{aligned}
&A \text{ is not } d\text{-closed} \iff \\
&A' \text{ is not } d\text{-open} \iff \\
&\exists\, x \in A' : \forall \varepsilon > 0, \mathcal{O}_d(x, \varepsilon) \bigcap A \neq \varnothing
\end{aligned}$$

For every $n \in \mathbb{N}$ choose an $x_n$ such that

$$x_n \in \mathcal{O}_d\left(x, \frac{1}{n+1}\right) \bigcap A$$

and hence construct a sequence in $A$.

Now consider a $\delta > 0$ such that for some $n_\delta$

$$\delta > \frac{1}{n_\delta + 1} \iff \frac{1}{\delta} < n_\delta + 1 \iff n_\delta < \frac{1-\delta}{\delta} < +\infty$$

so that

$$\forall \delta > 0 \ \exists \ n_\delta \in \mathbb{N} : \forall n \geq n_\delta, \ x_n \in \mathcal{O}_d\left(x, \frac{1}{n+1}\right) \subseteq \mathcal{O}_d(x, \delta)$$

Thus $x_n \to x$, but $x \notin A$, which leads to a contradiction. So, for a set $A \subseteq X$ such that for all sequences, $(x_n)_{n\in\mathbb{N}}$, in $A$ with $d - \lim x_n = x$ it holds that $x \in A$, it also has to hold that $A$ is a $d$-closed subset of $X$.

For the converse suppose that some $A \in X$ is $d$-closed and that $\exists \ (x_n)_{n\in\mathbb{N}}$ in $A$ with $x_n \to x$ and $x \notin A$. Then

$$\begin{aligned}
&A \text{ is } d\text{-closed} \iff \\
&A' \text{ is } d\text{-open} \iff \\
&\exists \ \varepsilon > 0 : \mathcal{O}_d(x, \varepsilon) \bigcap A = \varnothing
\end{aligned}$$

Since $x_n \to x$ almost every element of $(x_n)_{n\in\mathbb{N}}$ is in $\mathcal{O}_d(x, \varepsilon)$. But since this is a sequence in $A$ and $A$ and $\mathcal{O}_d(x, \varepsilon)$ are disjoint, this cannot hold. Contradiction! ■

So, we have found a way of characterizing $d$-closedness in metric spaces that is not in terms of $d$-openness (at least this is how it appears to be at a superficial level). One thing that we can do with this is “prove” that the empty set, $\varnothing$, is $d$-open and the carrier set, $X$, is $d$-closed. Consider a metric space, $(X, d)$, and all $d$-convergent sequences in it. Inevitably, all of the limits of these sequences will lie in $X$. Thus, by the above lemma, $X$ is a $d$-closed subset of itself. Consequently, $\varnothing$ is a $d$-open subset of $X$.

**Definition**

Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and $f$ a function from $X$ to $Y$, $f : X \to Y$. $f$ is $d_Y/d_X$-continuous at $x \in X$ iff

$$\forall (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ with } x = d_X - \lim(x_n) \Rightarrow f(x) = d_Y - \lim(f(x_n))$$

**Lemma**

Let $f : X \to Y$ be some function with $(X, d_X)$ and $(Y, d_Y)$ metric spaces. Then the following statements are all equivalent to one another (i.e. when any one of them holds, all them concurrently hold) for any point $x \in X$:

1. $f$ is $d_Y/d_X$-continuous at $x \in X$
2. $\forall \delta > 0, \exists \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$
3. If $A \in \tau_{d_Y}(f(x))$ then $\exists B \in \tau_{d_X}(x) : B \subseteq f^{-1}(A)$

**Proof**

We want to show that each of the above conditions is a *necessary* and *sufficient* condition for all of the others.

First, we show that 2. is a *necessary* and *sufficient* condition for 1.

Assume that 2. holds. Then $\forall \delta > 0, \exists\, \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$.

For $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $x_n, x \in X$ consider $(f(x_n))_{n\in\mathbb{N}}$. Then for some $\delta > 0$, choose a $\varepsilon_\delta$ that satisfies 2. Because $x_n \to x$

$$\forall n \geq n^*(\varepsilon_\delta),\, x_n \in \mathcal{O}_{d_X}(x, \varepsilon_\delta) \Rightarrow$$
$$\forall n \geq n^*(\varepsilon_\delta),\, f(x_n) \in f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right)$$

and because we assumed that $f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$

$$\forall n \geq n^*(\varepsilon_\delta),\ f(x_n) \in \mathcal{O}_{d_Y}(f(x), \delta)$$

and since $\delta$ is arbitrary $f(x_n) \to f(x)$. Because $(x_n)_{n\in\mathbb{N}}$ is arbitrary $f$ is $d_Y/d_X$-continuous at $x \in X$.

So 2. is a *sufficient* condition for 1. at $x$.

Now, suppose that 1. holds ($f$ is $d_Y/d_X$-continuous at $x \in X$), but for some $\delta > 0$ no $\varepsilon_\delta$ exists that satisfies 2. and $\forall \varepsilon > 0, f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x),\delta)$. This can be equivalently expressed as

$$\exists \delta > 0 : \forall \varepsilon > 0, f(\mathcal{O}_{d_X}(x,\varepsilon)) \bigcap \mathcal{O}'_{d_Y}(f(x),\delta) \neq \varnothing$$

This implies that

$$\exists \delta > 0 : \forall n \in \mathbb{N}, f(\mathcal{O}_{d_X}(x, \frac{1}{n+1})) \bigcap \mathcal{O}'_{d_Y}(f(x),\delta) \neq \varnothing$$

(since all $n \in \mathbb{N}$ give suitable $\varepsilon$). Consider the images of these sets through $f^{-1}$ (which are also non-empty)[2]

$$f^{-1}\left(f\left(\mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right)\right) \bigcap \mathcal{O}'_{d_Y}(f(x),\delta)\right) = \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right) \bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \neq \varnothing, \forall n \in \mathbb{N}$$

and a sequence, $(x_n)_{n\in\mathbb{N}}$, such that the $n$-th element of the sequence belongs to the $n$-th such set

$$x_n \in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right) \bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow$$
$$x_n \in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right)$$

which can be shown to imply a convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to $x$. Thus, $x = d_X - \lim(x_n)$.

By the assumed $d_Y/d_X$-continuity of $f$ at $x$, we get $f(x) = d_Y - \lim(f(x_n))$, but

$$x_n \in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right) \bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow$$
$$x_n \in f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \iff$$
$$f(x_n) \in f(f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))) = \mathcal{O}'_{d_Y}(f(x),\delta) \iff$$
$$f(x_n) \notin \mathcal{O}_{d_Y}(f(x),\delta)$$

which can be shown to make convergence of $(f(x_n))_{n\in\mathbb{N}}$ at $f(x) \in Y$ impossible. Hence we have a contradiction.

So 2. is a *necessary* condition for 1. at $x$.

Now, we show that 2. and 3. imply one another.

Let 3. hold.

[2]Even if $f$ is not one-to-one, we can think of $f^{-1}$ as a function $f^{-1} : Y \to P(X)$ that maps elements of $Y$ to the subsets of $X$ whose elements are mapped to the given value, i.e. $f^{-1}(y) = \{x \in X | f(x) = y\}$. Furthermore, we can think of $f$ and $f^{-1}$ as mappings between subsets of $X$ and $Y$, not just single elements, which is how we are using them here. In this case, the reverse image we are taking is not empty, because $f$ is used to produce the set we are passing to $f^{-1}$.

For $\delta > 0$ choose $A = \mathcal{O}_{d_Y}(f(x), \delta)$. By our assumption $\exists$ $B$ in the neighbourhood system $\tau_{d_X}(x)$ such that $B \subseteq f^{-1}(A)$. Since $B \in \tau_{d_X}(x)$ there always exists a $\varepsilon > 0$ such that $B$ is a subset of a $d_X$-open ball with center $x$ and radius $\varepsilon$. All this implies that

$$\begin{aligned} \mathcal{O}_{d_X}(x, \varepsilon) &\subseteq B \subseteq f^{-1}(A) \Rightarrow \\ \mathcal{O}_{d_X}(x, \varepsilon) &\subseteq f^{-1}(\mathcal{O}_{d_Y}(f(x), \delta)) \Rightarrow \\ f(\mathcal{O}_{d_X}(x, \varepsilon)) &\subseteq f(f^{-1}(\mathcal{O}_{d_Y}(f(x), \delta))) \Rightarrow \\ f(\mathcal{O}_{d_X}(x, \varepsilon)) &\subseteq \mathcal{O}_{d_Y}(f(x), \delta) \end{aligned}$$

So 3. implies 2.

Now, let 2. hold.

Suppose that $\exists A \in \tau_{d_Y}(f(x))$ such that $\forall B \in \tau_{d_X}(x)$, $B$ is not a subset of $f^{-1}(A)$, i.e.

$$B \bigcap \left(f^{-1}(A)\right)' \neq \varnothing \qquad \forall B \in \tau_{d_X}(x)$$

Because all $d_X$-open balls with center $x$ belong to $\tau_{d_X}(x)$

$$\begin{aligned} \mathcal{O}_{d_X}(x, \varepsilon) \bigcap \left(f^{-1}(A)\right)' &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ \mathcal{O}_{d_X}(x, \varepsilon) \bigcap f^{-1}(A') &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon) \bigcap f^{-1}(A')\right) &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap f\left(f^{-1}(A')\right) &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap A' &\neq \varnothing & \forall \varepsilon > 0 \end{aligned}$$

But $A \in \tau_{d_Y}(f(x))$ so there always exists $\delta > 0 : \mathcal{O}_{d_Y}(f(x), \delta) \subseteq A$. This implies that $(\mathcal{O}_{d_Y}(f(x), \delta))' \supseteq A'$ and thus

$$f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap (\mathcal{O}_{d_Y}(f(x), \delta))' \neq \varnothing \qquad \forall \varepsilon > 0$$

which is equivalent to $f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x), \delta)$ and contradicts 2.

So 2. implies 3. ■