

Problem Set 2

Open and Closed Balls, Boundness, and Total Boundness

Exercise 1

Show that open and closed balls can be defined for pseudo-metric spaces.

Exercise 2

For the following (X, d) pairs, show that they constitute (pseudo-)metric spaces and define the unit open balls on them and visualize them:

1. $X = \mathbb{R}$ and $d : X \times X \rightarrow \mathbb{R}$, such that

$$d(x, y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}, \forall x, y \in X$$

with $c > 0$.

2. $X = \mathbb{R}^2$ and $d : X \times X \rightarrow \mathbb{R}$, such that

$$d(x, y) = \sqrt{(x - y)' A (x - y)}, \forall x, y \in X$$

with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

3. $X = \mathbb{R}^3$ and $d : X \times X \rightarrow \mathbb{R}$, such that

$$d(x, y) = \max \left\{ |x_1 - y_1|, \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2} \right\}, \forall x, y \in X$$

*Please report any typos, mistakes, or even suggestions at zaverdasd@aueb.gr.

Exercise 3

Let (X, d) be a metric space and for some $x, y \in X$ and $\varepsilon > 0$ let $y \in \mathcal{O}_d(x, \varepsilon)$. Show that $\exists \delta > 0 : \mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$.

Exercise 4

Let (X, d) be a metric space and $Y \subseteq X$. Define $d' : Y \times Y \rightarrow \mathbb{R}$ such that $d' = d|_{Y \times Y}$. Then (Y, d') is a metric subspace of (X, d) . Show that $\mathcal{O}_{d'}(x, \varepsilon) = \mathcal{O}_d(x, \varepsilon) \cap Y$ and $\mathcal{O}_{d'}[x, \varepsilon] = \mathcal{O}_d[x, \varepsilon] \cap Y$.

Exercise 5

Is $(0, 1)$ a bounded set?

Exercise 6

Let $d : \mathbb{R}_{++} \times \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that $d(x, y) = |\ln(x) - \ln(y)|$, $\forall x, y \in \mathbb{R}_{++}$ be a metric on \mathbb{R}_{++} . Is $(0, 1)$ a d -bounded subset of \mathbb{R}_{++} ?

Exercise 7

Let (Y, d) be a metric space and $X \neq \emptyset$. For $d_{sup} : \mathcal{B}(X, Y) \times \mathcal{B}(X, Y) \rightarrow \mathbb{R}$ such that

$$d_{sup}(f, g) = \sup_{x \in X} d(f(x), g(x)), \forall f, g \in \mathcal{B}(X, Y)$$

show that:

1. $(\mathcal{B}(X, Y), d_{sup})$ is a metric space.
2. If (Y, d) is bounded, then $(\mathcal{B}(X, Y), d_{sup})$ is also bounded.

Exercise 8

Let $X \subseteq \mathbb{R}^N$ with $N \in \mathbb{N}^*$ and $d : X \times X \rightarrow \mathbb{R}$ such that $d(x, y) = \left(\sum_{i=1}^N |x_i - y_i|^2 \right)^{\frac{1}{2}}$, $\forall x, y \in X$ be the Euclidean metric on X . Show that d -boundedness in X is sufficient for d -total boundedness in X .

(Hint: Consider a d -bounded set in X and show that the ball that covers it is d -totally bounded.)

Exercise 9

Let $X = \{(x_n)_{n \in \mathbb{N}^*} : x_n \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 < +\infty\}$ (i.e. X is the set of square summable real sequences) and $d : X \times X \rightarrow \mathbb{R}$ such that $d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2 \right)^{\frac{1}{2}}$, $\forall x, y \in X$ be a metric on X . Show that d -boundedness in X is not sufficient for d -total boundedness in X .

(Hint: Consider the sequence $\mathbf{0} = \{0\}_{n \in \mathbb{N}^*} \in X$ and the d -closed unit ball centered at it. Use Riesz's Lemma and the Pigeonhole Principle.)

Exercise 10

Let $X = \{f : [0, 1] \rightarrow \mathbb{R} \mid \int_0^1 f^2(x)dx < +\infty\}$ be the set of square integrable functions from $[0, 1]$ to \mathbb{R} . Consider the metric function $d(f, g) := \left(\int_0^1 (f(x) - g(x))^2 dx \right)^{\frac{1}{2}}$ on X . If $\mathbf{0} : [0, 1] \rightarrow \mathbb{R}$ is a function in X such that $\mathbf{0}(x) := 0 \forall x \in [0, 1]$, consider $\mathcal{O}_d[\mathbf{0}, 1]$ and show that it is not d -totally bounded.

Exercise 11

Let (X_i, d_i) be metric spaces $\forall i \in \mathcal{I}$ with \mathcal{I} a finite index set. For the cartesian product $X := \prod_{i \in \mathcal{I}} X_i$ there can be defined the following structured sets (X, d_Π) with $d_\Pi \in \{d_{\Pi_{max}}, d_{\Pi_I}, d_{\Pi_{| |}}\}$ and d_Π are defined as

$$\begin{aligned} d_{\Pi_{max}} &= \max_{i \in \mathcal{I}} d_i \\ d_{\Pi_I} &= \left(\sum_{i \in \mathcal{I}} d_i^2 \right)^{\frac{1}{2}} \\ d_{\Pi_{| |}} &= \sum_{i \in \mathcal{I}} d_i \end{aligned}$$

and are appropriate metric functions on X . Let $A_i \subseteq X_i, \forall i \in \mathcal{I}$ and $A := \prod_{i \in \mathcal{I}} A_i$, which implies that $A \subseteq X$. Show that A is a d_Π -totally bounded subset of X iff A_i are d_i -totally bounded subsets of $X_i \forall i \in \mathcal{I}$, for each of the three d_Π defined above.

Exercise 12

Let d_1 and d_2 be both metrics on a non empty set X such that $d_1 \leq cd_2$ with $c > 0$. Show that:

1. for $x \in X$ and $0 < \varepsilon$ then $\mathcal{O}_{d_2}(x, \varepsilon) \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon)$.
2. if $A \subseteq X$ is d_2 -bounded, then it is d_1 -bounded.
3. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1 -bounded iff it is d_2 -bounded.
4. if $A \subseteq X$ is d_2 -totally bounded, then it is d_1 -totally bounded.
5. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1 -totally bounded iff it is d_2 -totally bounded.

Useful Theorems and Results

Diagonal of a Euclidean N -cube

Let C be a “cube” in a Euclidean space with side length $\alpha > 0$. That is, if $x \in \mathbb{R}^N$ is the “center” of C , then

$$C = \left[x_1 - \frac{\alpha}{2}, x_1 + \frac{\alpha}{2} \right] \times \left[x_2 - \frac{\alpha}{2}, x_2 + \frac{\alpha}{2} \right] \times \dots \times \left[x_N - \frac{\alpha}{2}, x_N + \frac{\alpha}{2} \right]$$

Then the maximum distance from this center x is equal to

$$\begin{aligned} \max_{y \in C} d(x, y) &= \max_{y \in C} \sqrt{\sum_{i=1}^N |x_i - y_i|^2} \\ &= \max_{\{y_i \in [x_i - \frac{\alpha}{2}, x_i + \frac{\alpha}{2}]\}_{i=1}^N} \sqrt{\sum_{i=1}^N |x_i - y_i|^2} \\ &= \sqrt{\sum_{i=1}^N \left| x_i - x_i \pm \frac{\alpha}{2} \right|^2} \\ &= \sqrt{\sum_{i=1}^N \left| \pm \frac{\alpha}{2} \right|^2} \\ &= \frac{\alpha}{2} \sqrt{\sum_{i=1}^N 1} \\ &= \frac{\alpha}{2} \sqrt{N} \end{aligned}$$

and corresponds to all the “corners” of this N -cube.

Riesz’s Lemma

For (X, d) normed vector space (i.e. the metric d is a p -norm), $(S, d|_{S \times S})$ non-dense linear subspace of (X, d) , and $0 < \varepsilon < 1$, there exists $x \in X$ of unit norm (i.e. $d(\mathbf{0}, x) = \|x\|_p = 1$) such that $d(x, s) \geq 1 - \varepsilon, \forall s \in S$.

Pigeonhole Principle

For $n, m, k \in \mathbb{N}$ with $n = km + 1$, if we distribute n elements across m sets then at least one set will contain at least $k + 1$ elements.