Athens University of Economics and Business
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Postgraduate Program - MSc in Economic Theory
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Completeness and Function Spaces

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Lemma

Let (Y, d) be a complete metric space and $X \neq \varnothing$ a non-empty set, then the structured set $(\mathcal{B}(X,Y), d^d_{sup})$ with $d^d_{sup}(f,g) = \sup_{x \in X} d(f(x), g(x))$, $\forall f, g \in \mathcal{B}(X,Y)$ is a complete metric space.

Proof

We need to show that:

1. $(\mathcal{B}(X,Y), d^d_{sup})$ is a metric space, which requires that:
   - $\mathcal{B}(X,Y)$ be a non-empty set.
   - d^d_{sup} be a metric function on $\mathcal{B}(X,Y)$ (properties i-iv).
2. Every d^d_{sup}-Cauchy sequence on $\mathcal{B}(X,Y)$ has a d^d_{sup}-limit in $\mathcal{B}(X,Y)$, which requires that for all $(f_n)_{n \in \mathbb{N}} : f_n \in \mathcal{B}(X,Y)$, $\forall n \in \mathbb{N}$, there exists a function, f, such that:
   - $f = d^d_{sup} - \lim f_n$
   - $f \in \mathcal{B}(X,Y)$

1. Since $X \neq \varnothing$ we can define at least one function (if not more) that maps elements of X to elements of Y (also non-empty). For example, the constant function $f_c : X \to Y$ such that $\forall x \in X, f_c(x) = y_c$ for some $y_c \in Y$.

   Furthermore, observe that $f_c(X) = \{y_c\} \subseteq Y$, i.e. the image of X through f_c is a subset of Y (naturally) and it is also a singleton set (it has only one element). Thus $f(X)$ is certainly a d-bounded subset of Y.

   We have found at least one example of a bounded function from X to Y. So $\mathcal{B}(X,Y)$ is non-empty.

   Also, d^d_{sup} is a metric function on $\mathcal{B}(X,Y)$ since:

   i) $\forall f, g \in \mathcal{B}(X,Y)$, $d^d_{sup}(f,g) = \sup_{x \in X} d(f(x), g(x)) \overset{i}{\geq} 0$

   ii) $\forall f, g \in \mathcal{B}(X,Y)$,

$$
\begin{aligned}
d^d_{sup}(f,g) = 0 & \iff \\
\sup_{x \in X} d(f(x), g(x)) = 0 & \overset{i}{\iff} \\
d(f(x), g(x)) = 0,\ \forall x \in X & \overset{ii}{\iff} \\
f(x) = g(x),\ \forall x \in X & \iff \\
f = g &
\end{aligned}
$$

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iii) $\forall f, g \in \mathcal{B}(X,Y)$, $d^d_{sup}(f,g) = \sup\limits_{x\in X} d(f(x), g(x)) \overset{iii}{=} \sup\limits_{x\in X} d(g(x), f(x)) = d^d_{sup}(g,f)$

iv) $\forall f, g, h \in \mathcal{B}(X,Y)$,

$$\begin{aligned} d^d_{sup}(f,g) &= \sup_{x\in X} d(f(x), g(x)) \\ &\leq \sup_{x\in X} \left(d(f(x), h(x)) + d(h(x), g(x))\right) \\ &\leq \sup_{x\in X} d(f(x), h(x)) + \sup_{x\in X} d(h(x), g(x)) \\ &= d^d_{sup}(f,h) + d^d_{sup}(h,g) \end{aligned}$$

So d^d_{sup} is a metric function on the non-empty $\mathcal{B}(X,Y)$ and $(\mathcal{B}(X,Y), d^d_{sup})$ is a metric space.

2. Consider an arbitrary d^d_{sup}-Cauchy sequence on $\mathcal{B}(X,Y)$, $(f_n)_{n\in\mathbb{N}} : f_n \in \mathcal{B}(X,Y)\, \forall n \in \mathbb{N}$. Then $\forall \varepsilon > 0\ \exists\ n(\varepsilon)$ such that

$$\begin{aligned} d^d_{sup}(f_n, f_m) &< \varepsilon,\ \forall n, m > n(\varepsilon) \\ \sup_{x\in X} d(f_n(x), f_m(x)) &< \varepsilon,\ \forall n, m > n(\varepsilon) \\ \forall x \in X,\ d(f_n(x), f_m(x)) &< \varepsilon,\ \forall n, m > n(\varepsilon) \end{aligned}$$

so that $(f_n(x))_{n\in\mathbb{N}}$ is a d-Cauchy sequence on Y, $\forall x \in X$.

Here is an informal breakdown to facilitate your intuition. From one d^d_{sup}-Cauchy sequence in the set of bounded Y-valued functions, $\mathcal{B}(X,Y)$, we can get multiple sequences, $(y_n)_{n\in\mathbb{N}}$, in Y that are d-Cauchy, one for every $x \in X$. For each such starting sequence, $(f_n)_{n\in\mathbb{N}}$, there are as many such $(y_n)_{n\in\mathbb{N}}$ sequences as elements in the domain of those f_n, X, and their n-th element is given by $y_n = f_n(x)$.

Furthermore, because (Y, d) is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges to a d-limit in Y, say $\phi_x \in Y$, for all $x \in X$.

So starting from a d^d_{sup}-Cauchy sequence on $\mathcal{B}(X,Y)$ we can generate a collection of d-convergent sequences on Y (one sequence for each $x \in X$). That is, we have that

$$\begin{aligned} &\forall\ (f_n)_{n\in\mathbb{N}} : (f_n)_{n\in\mathbb{N}}\ d^d_{sup}\text{-Cauchy on } \mathcal{B}(X,Y) \\ &\exists\ \{(y_n)_{n\in\mathbb{N}} : y_n = f_n(x)\, \forall n \in \mathbb{N}, (y_n)_{n\in\mathbb{N}}\ d\text{-convergent on } Y,\ \forall x \in X\} \end{aligned}$$

Now, for any such starting sequence $(f_n(x))_{n\in\mathbb{N}}$ using the corresponding ϕ_x of each $x \in X$, define the function $f : X \to Y$ such that $f(x) \coloneqq \phi_x$, $\forall x \in X$. To show that $(\mathcal{B}(X,Y), d^d_{sup})$ is complete, it suffices to show that $d^d_{sup} - \lim f_n = f$ and that $f \in \mathcal{B}(X,Y)$ (since we have taken $(f_n(x))_{n\in\mathbb{N}}$ to be d-Cauchy).

Firstly, notice that

$$\begin{aligned} d^d_{sup}(f_n, f) &= \sup_{x\in X} d(f_n(x), f(x)) \\ &= \sup_{x\in X} d(f_n(x),\ d - \lim_{m\to+\infty} f_m(x)) \\ &= \sup_{x\in X} \lim_{m\to+\infty} \Big(d(f_n(x), f_m(x)) \Big) \end{aligned}$$

This last equality follows by the fact that metrics are continuous functions (*needs proof*).

Now consider $d(f_n(x), f_m(x))$ for some $x \in X$ (notice that it is a real number). Choose an arbitrary fixed $n \in \mathbb{N}$ and think of the real valued sequence $(z_m)_{m\in\mathbb{N}} \in \mathbb{R}$, such that $z_m = d(f_n(x), f_m(x))$ for any given $n \in \mathbb{N}$ and $x \in X$.

$$\begin{aligned} d^d_{sup}(f_n, f) &= \sup_{x\in X} \lim_{m\to+\infty} z_m \\ &\leq \sup_{x\in X} \sup_{m\geq n} z_m \end{aligned}$$

This holds because we are considering more z_m than just those that "tend to infinity". Keep in mind that, if we denote z as the limit here, even if $z_m < z$, $\forall m \in \mathbb{N}$ it still holds that $\sup_{m\in\mathbb{N}} z_m = z$. This can be generalized to cases where a finite number of z_m are greater than the limit (if they were infinite, then it wouldn't be a limit).

So we have that

$$\begin{aligned} d^d_{sup}(f_n, f) &\leq \sup_{x\in X} \sup_{m\geq n} d(f_n(x), f_m(x)) \\ &\leq \sup_{x\in X} \sup_{m\geq n} \sup_{x\in X} d(f_n(x), f_m(x)) \\ &= \sup_{m\geq n} \sup_{x\in X} d(f_n(x), f_m(x)) \\ &= \sup_{m\geq n} d^d_{sup}(f_n, f_m) \end{aligned}$$

So we showed that $d^d_{sup}(f_n, f) \leq \sup_{m\geq n} d^d_{sup}(f_n, f_m)$. Now consider the following

$$\begin{aligned} &(f_n)_{n\in\mathbb{N}} \text{ is } d^d_{sup}\text{-Cauchy} \\ &\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : d^d_{sup}(f_n, f_m) < \varepsilon && \forall n, m \geq n(\varepsilon) \\ &\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : \sup_{m\geq n} d^d_{sup}(f_n, f_m) < \varepsilon && \forall n \geq n(\varepsilon) \\ &\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : d^d_{sup}(f_n, f) < \varepsilon && \forall n \geq n(\varepsilon) \end{aligned}$$

thus as $n \to +\infty$, $f_n \to f$ with respect to d^d_{sup}.

And because $(f_n(x))_{n\in\mathbb{N}}$ was arbitrarily chosen, any d^d_{sup}-Cauchy sequence in $\mathcal{B}(X, Y)$ is d^d_{sup}-convergent. It still remains to show that its d^d_{sup}-limit (say f) is in $\mathcal{B}(X, Y)$.

So, for any two points $f(x), f(y) \in Y$ and some $n \in \mathbb{N}$ we have that

$$\begin{aligned}
\sup_{x,y\in X} d(f(x), f(y)) &\leq \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f(y)) \\
&\leq \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&\leq \sup_{x\in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{y\in X} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&= 2\sup_{x\in X} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&= 2d^d_{sup}(f_{n(\varepsilon)}, f) + \sup_{x,y\in X} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y))
\end{aligned}$$

where $n(\varepsilon)$ is such that $d^d_{sup}(f_{n(\varepsilon)}, f) < \varepsilon$. This $n(\varepsilon)$ exists since $f_n \to f$ with respect to d^d_{sup}. Thus, the first additive term is bounded by 2ε. The second additive term is the maximum distance between all values of $f_{n(\varepsilon)}$ on Y. Since $f_{n(\varepsilon)} \in \mathcal{B}(X, Y)$ this number is also bounded. Thus, $\sup_{x,y\in X} d(f(x), f(y)) < +\infty$, which establishes that $f \in \mathcal{B}(X, Y)$, i.e. f is a bounded Y-valued function.

So for (Y, d) complete metric space, every d^d_{sup}-Cauchy sequence on $\mathcal{B}(X, Y)$ is d^d_{sup}-convergent in $\mathcal{B}(X, Y)$.

Thus, if (Y, d) is a complete metric space, then $(\mathcal{B}(X, Y), d^d_{sup})$ is also a complete metric space.