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Introduction to some Topological Notions

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Basic Calculus and Real Analysis have shaped our intuition on what is the distance between two real numbers and more generally elements of $\mathbb{R}^N$ with $N \in \mathbb{N}^*$. Let $x, y \in \mathbb{R}$, then we instinctively say that their distance is simply their absolute difference, $|x - y|$. Let $x, y \in \mathbb{R}^2$, then their distance is given by applying the Pythagorean Theorem, $\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, where $x = (x_1,\, x_2)$ and $y = (y_1,\, y_2)$.

The study of metric spaces has shown us that we can generalize the idea of the distance between elements of $\mathbb{R}$ or $\mathbb{R}^N$. We realize that the above mentioned absolute distance or the Euclidean distance are just specific metric functions that we can pair with the appropriate set. Furthermore, we could have used other appropriate functions to define the notion of distance between their elements. We also realize that we can define distance for different kinds of sets, possibly much more "exotic" than subsets of the real numbers and their cartesian products, by pairing them with appropriate metric functions which satisfy a list of desired properties on the set in question. We call such pairings *metric spaces* and they are comprised of a non-empty *carrier set* and a well behaved *metric function*.

General Topology abstracts away from the idea of a numerically quantifiable distance and replaces *metric functions* with so called *topologies* on the carrier set. Such pairings are called *topological spaces*.

Here we will first be introduced to d-openness and d-closedness in metric spaces. Next, we will define topologies, τ, which are sets of subsets of the carrier set. A subset of the carrier set is considered to be open with respect to τ, by merit of being an element of τ (not to be confused with d-openness with respect to some metric, d). We will then return to metric spaces and talk about sequential convergences and continuity and finally prove a lemma that shows that we can

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equivalently characterize the continuity of a function in terms of both metric spaces and topological spaces of its domain and co-domain.

Openness (and closedness) in metric spaces

Definition

Let (X, d) be a metric space and A a subset of X (not necessarily non-empty). A is a d-open subset of X (i.e. open with respect to the metric d) iff

$$\forall x \in A, \exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

The x subscript on ε_x means that generally this radius may depend on the element in question.

For any metric space, (X, d), there is a number of examples of d-open subsets of the carrier set, X. First, there is the carrier set, X, itself, because for every one of its elements we can find a radius such that the constructed d-open ball is a subset of the carrier set (and in fact any radius will do). For now, we will axiomatically say that the empty set, $\varnothing$, is also a d-open subset of the carrier set of X, but we will also show why that is when we will talk about sequential d-convergence as a way of showing d-closedness. Finally, it can be shown that d-open balls are also d-open subsets of their carrier sets.

Lemma

Let (X, d) be a metric space. Any d-open ball in it is a d-open subset of X.

Proof

Choose an arbitrary $x \in X$ and radius $\varepsilon > 0$ and define the d-open ball $\mathcal{O}_d(x, \varepsilon)$.
For some $y \in \mathcal{O}_d(x, \varepsilon)$ define $\delta \coloneqq \varepsilon - d(x, y) > 0$.
Then $\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$ (to see why see Problem Set 2 Exercise 3).
So $\mathcal{O}_d(x, \varepsilon)$ is a d-open subset of X. Since x and ε were chosen arbitrarily, every d-open ball is a d-open of subset of X. ■

Furthermore, the following properties hold with respect to d-openness:

- Arbitrary unions of d-open sets are d-open.

  It is easy to intuitively understand why. If for an element of a d-open set, $x \in A$, we can find a radius, $\varepsilon_x > 0$, such that $\mathcal{O}_d(x, \varepsilon_x)$ lies entirely in A, then this would hold for any arbitrary union of A with other sets. This holds for any $x \in A$ and if all sets in the union are d-open, then the union is d-open.

- Finite intersections of d-open sets are d-open.

  To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric, d_u (absolute difference).

  Let A_n be subsets of $\mathbb{R}$ such that

$$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \forall\, n \in \mathbb{N}^*$$

  With respect to the usual metric on $\mathbb{R}$, all A_n are d_u-open subsets of $\mathbb{R}$.

  Now notice that there is an infinite number of such sets and their only common element is 0. So their intersection (an infinite intersection of d_u-open sets) is $\{0\}$.

  But also notice that $\forall \varepsilon > 0\ \mathcal{O}_{d_u}(0, \varepsilon) \supset \{0\}$. So $\nexists\, \varepsilon > 0 : \mathcal{O}_{d_u}(0, \varepsilon) \subseteq \{0\}$.

  Thus we have found an infinite intersection of d_u-open sets that is not d_u-open.

Definition

Let (X, d) be a metric space. Any subset, A, of X is termed *a d-closed subset of* X if its complement, A' (i.e. $A' = X \setminus A$), is a d-open subset of X.

This definition has a few implications. First, the carrier set, X, is a d-closed subset of itself, since the empty set is a d-open subset of X. Secondly, the empty set, $\varnothing$, is a d-closed subset of X, since X is a d-open subset of itself. Finally, it can be shown that d-closed balls are also d-closed subsets of X.

Lemma

Let (X, d) be a metric space. Any d-closed ball in it is a d-closed subset of X.

Proof

Consider an arbitrary d-closed ball in X, $\mathcal{O}_d[x, \varepsilon]$. It suffices to show that its complement, $\mathcal{O}'_d[x, \varepsilon]$, is a d-open subset of X.

Let $y \in \mathcal{O}'_d[x, \varepsilon] \iff d(x, y) > \varepsilon$. We want to find a radius, δ, such that

$$\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}'_d[x, \varepsilon]$$
$$\mathcal{O}_d(y, \delta) \bigcap \mathcal{O}_d[x, \varepsilon] = \varnothing$$
$$d(x, z) > \varepsilon, \; \forall \; z \in \mathcal{O}_d(y, \delta)$$

Now, define $\delta := d(x, y) - \varepsilon > 0$ and let $z \in \mathcal{O}_d(y, \delta)$. Naturally, the dual of the triangle inequality holds between the chosen x, y, and z (see Problem Set 1 Exercise 5)

$$d(x, z) \geq |d(x, y) - d(y, z)|$$
$$d(x, z) \geq |\varepsilon + \delta - d(y, z)|$$
$$d(x, z) \geq \varepsilon + |\delta - d(y, z)|$$
$$d(x, z) > \varepsilon$$

Since z was chosen arbitrarily, the above holds for all $z \in \mathcal{O}_d(y, \delta)$. So there exists a δ for y such that $d(x, z) > \varepsilon$, $\forall z \in \mathcal{O}_d(y, \delta)$.

Since y was chosen arbitrarily, the above holds for all $y \in \mathcal{O}'_d[x, \varepsilon]$. So $\mathcal{O}'_d[x, \varepsilon]$ is a d-open subset of X and $\mathcal{O}_d[x, \varepsilon]$ is d-closed.

Since $\mathcal{O}_d[x, \varepsilon]$ was chosen arbitrarily, the above holds for every d-closed ball. ■

Furthermore, the following properties hold with respect to d-closedness:

- Finite unions of d-closed sets are d-closed.

  To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric, d_u.

  Let A_n be subsets of $\mathbb{R}$ such that

  $$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \forall \; n \in \mathbb{N}^*$$

  All A_n are d_u-closed subsets of $\mathbb{R}$.

  Now notice that there is an infinite number of such sets and that

  $$\bigcup_{n=2}^{\infty} A_n = (0, 1)$$

So their union (an infinite union of d_u-closed sets) is $B := (0, 1)$.

But also notice that $B' = (-\infty, 0] \cup [1, +\infty)$ and that for its elements 0 and 1 there do not exist a radii $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that $\mathcal{O}_{d_u}(0, \varepsilon_0) \subseteq B'$ and $\mathcal{O}_{d_u}(1, \varepsilon_1) \subseteq B'$, respectively. So B' is not an d_u-open subset of $\mathbb{R}$ and its complement B, an infinite union of d_u-closed sets, is not d_u-closed.

- Arbitrary intersections of d-closed sets are d-closed.

  To intuitively see why, first consider a d-closed subset of some carrier set. Then its complement is d-open. Now notice that the complement of the intersection of two sets is the union of their complements. The complement of an arbitrary intersection of d-closed sets is the union of the arbitrary complements, which are d-open. So the union is d-open and the intersection is d-closed.

It is important to realize that d-openness and d-closedness are not necessarily mutually exclusive, nor are they opposites. A set can be d-open, d-closed, both d-open and d-closed, or neither d-open or d-closed.

For example, consider the set of real numbers endowed with the usual metric, d_u. By our intuition (but it can also be shown rigorously) $(0, 1)$ is a d_u-open subset of $\mathbb{R}$ but not d_u-closed. $[0, 1]$ is a d_u-closed subset of $\mathbb{R}$ but not d_u-open. $[0, 1)$ is neither a d_u-open nor a d_u-closed subset of $\mathbb{R}$. Finally, endow $\mathbb{R}$ with the discrete metric, $d_\delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

$$d_\delta(x, y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

then $\{0\}$ is a d_δ-open subset of $\mathbb{R}$, because $\mathcal{O}_{d_\delta}\left(0, \frac{1}{2}\right) \subseteq \{0\}$ and it is also d_δ-closed because $\{0\}' = \mathbb{R}^*$ is a d_δ-open subset of $\mathbb{R}$ (because $\mathcal{O}_{d_\delta}\left(x, \frac{1}{2}\right) \subseteq \mathbb{R}^*$, $\forall x \in \mathbb{R}^*$).

Topologies

Definition

A topology, τ, on a set, X, is a collection of subsets of X that satisfy the following properties:

1. $\varnothing$ and X belong to τ ($\varnothing, X \in \tau$)

2. any *arbitrary* union of elements of τ is also an element of τ (τ is closed with respect to arbitrary unions) ($A_i \in \tau$, $\forall\, i \in \mathcal{I} \Rightarrow \bigcup_{i\in\mathcal{I}} A_i \in \tau$)

3. any *finite* intersection of elements of τ is also an element of τ (τ is closed with respect to finite intersections) ($A_i \in \tau$, $\forall\, i \in \mathcal{I}$ finite $\Rightarrow \bigcap_{i\in\mathcal{I}} A_i \in \tau$)

The pair (X, τ) is termed *topological space.*

In a topological space, (X, τ), any subset of X, $A \subseteq X$, that also belongs to the chosen topology, τ, ($A \subseteq \tau$) is by definition an open subset of X (and A', its complement in X, is closed). A more vague definition of a topology could be that it is *a collection of "open" subsets of* X. Notice that we are no longer talking about d-openness, because openness is no longer determined by a metric function, but by the chosen topology, τ.

Additionally, a metric, d, on X can be also used to generate a topology, τ_d, on X.

Lemma

For every metric function, d, that a non-empty set, X, is endowable with, there exists an implied topology, τ_d, where

$$\tau_d = \{A \subseteq X : A \text{ is } d\text{-open}\}$$

Proof

For τ_d to be a valid topology on X, it has to satisfy the properties 1-3 of topologies. The definition of τ_d basically says that every d-open subset of X belongs to it. So

1. $\varnothing$ and X belong to τ_d because they are d-open subsets of X.

2. Since all elements of τ_d are d-open subsets of X and arbitrary unions of d-open sets are d-open, then arbitrary unions of elements of τ_d also belong to τ_d.

3. Since all elements of τ_d are d-open subsets of X and finite intersections of d-open sets are d-open, then finite intersections of elements of τ_d also belong to τ_d.

Thus, τ_d is a topology on X. ■

An example of a topology generated by a metric is the discrete topology, τ_δ, which is generated by the discrete metric, d_δ.

Let (X, d_δ) be a metric space and A an arbitrary subset of X. Then for any element, x, in A there exists a positive radius that is less than one, $0 < \varepsilon < 1$, such that $\mathcal{O}_{d_\delta}(x, \varepsilon) = \{x\} \subseteq A$. So A is d_δ-open and since it was chosen arbitrarily, every subset of X is open with respect to the discrete metric.

So the discrete topology of a set is its powerset, $\tau_\delta = P(X)$.

However, not all topologies can be generated by a metric. One such example can be the indiscrete topology, $\tau_I = \{\varnothing, X\}$, on a non singleton set. Let $X = \{a, b\}$ with $a \neq b$, and assume that there exists a metric d that we can endow X with and that it can produce the indiscrete topology, τ_I. So we want $\exists\, d \mid \tau_I \equiv \{x \in X : x \text{ is } d\text{-open}\}$.

Define $\varepsilon := d(a, b)$ and the d-open ball $\mathcal{O}_d(a, \varepsilon)$.

Notice that $\mathcal{O}_d(a, \varepsilon) \in \tau_I$, since it is a d-open subset of X and τ_I is the collection of d-open subsets of X (since it is generated by d).

Also, $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$ since open balls always contain their center.

Finally, $\mathcal{O}_d(a, \varepsilon) \neq X$ since it does not include b.

Thus, $\mathcal{O}_d(a, \varepsilon) \in \tau_I = \{\varnothing, X\}$ and $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$ and $\mathcal{O}_d(a, \varepsilon) \neq X$. Contradiction!

So τ_I cannot always be generated by a metric. We can, thus, say that *not all topological spaces are metrizable.*

Finally, not all collections of subsets constitute topologies. To see this, let $X = \{a, b, c\}$ and a collection of subsets, $\tau = \{\varnothing, \{a, b\}, \{b, c\}, X\}$. Then the intersection between $\{a, b\}$ and $\{b, c\}$ is $\{b\} \notin \tau$. So τ violates property 3 and is not a topology on X.

We can also dually define a topology, τ_d^*, such that $\tau_d^* = \{A \subseteq X, A \text{ is } d\text{-closed}\}$ and it contains the same informational context as τ_d. Contrary to τ_d, we want τ_d^* to be closed under *finite* unions and *arbitrary* intersections of its elements.

Definition

Let (X, τ) be a topological space and $x \in X$, then

$$\tau(x) = \{A \in \tau, x \in A\}$$

is called a *neighbouring system of x.*

Convergence and Continuity in Metric Spaces

Definition

Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then we say that $x \in X$ is a d-limit of $(x_n)_{n\in\mathbb{N}}$ iff

$$\forall\, \varepsilon > 0,\, \exists\, n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \quad x_n \in \mathcal{O}_d(x, \varepsilon)$$

and can denote $x = d - \lim(x_n)$ and equivalently write $x_n \to x$ (x_n tends to x).

Lemma

Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $x \in X$. Consider τ_d as the topology on X generated by d. Then $x_n \to x$ iff $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A.

Proof

Suppose that x is the d-limit of $(x_n)_{n\in\mathbb{N}}$.

Choose an arbitrary $A \in \tau_d(x) \subseteq \tau_d$. By definition A is a d-open subset of X and $x \in A$. So

$$\exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

But because $x_n \to x$

$$\exists\, n_{\varepsilon_x} \in \mathbb{N} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

i.e. almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{O}_d(x, \varepsilon_x)$ and since $\mathcal{O}_d(x, \varepsilon_x) \subseteq A$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A. Since A was chosen arbitrarily, this holds for all such $A \subseteq \tau_d(x)$.

For the converse, suppose that $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A.

Since $\mathcal{O}_d(x, \varepsilon)$ is a d-open subset of X for all $\varepsilon > 0$ and it includes x, it follows that $\mathcal{O}_d(x, \varepsilon) \in \tau_d(x)$, which drives the result. ■

Lemma

Let (X, d) be a metric space. Every d-convergent sequence in X has a unique d-limit.

Proof

Let $(x_n)_{n\in\mathbb{N}}$ be a d-convergent sequence in X with $x = d - \lim(x_n)$, $y = d - \lim(x_n)$, and $x \neq y$. By separateness we know that $d(x, y) > 0$ so there exist $0 < \varepsilon_x < d(x, y)$ and $0 < \varepsilon_y < d(x, y)$ such that

$$\mathcal{O}_d(x, \varepsilon_x) \bigcap \mathcal{O}_d(y, \varepsilon_y) = \varnothing$$

Since $x_n \to x$ for this particular ε_x

$$\exists\, n_{\varepsilon_x} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

thus only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(x, \varepsilon_x)$.
Analogously, since $x_n \to y$ only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(y, \varepsilon_y)$.
But $\mathcal{O}_d(x, \varepsilon_x)$ and $\mathcal{O}_d(y, \varepsilon_y)$ are disjoint sets. Contradiction! ■

Observe that we need the separateness property to prove the uniqueness of d-limits. Thus, d-limits are not necessarily unique in pseudometric spaces.

Furthermore, the uniqueness of limits cannot necessarily be generalized in topological spaces that are not generated by a metric. Consider the example of the indiscrete topological space where the carrier set is not a singleton, e.g. (X, τ_I) with $X = \{a, b\}$, $a \neq b$, and $\tau_I = \{\varnothing, X\}$. Consider any sequence, $(x_n)_{n\in\mathbb{N}}$, in X. Here, a is a limit of $(x_n)_{n\in\mathbb{N}}$ because the neighbouring system of a with respect to τ_I comprises only of X itself

$$\tau_I(a) = \{X\}$$

and the entire sequence is in X.

Similarly, $\tau_I(b) = \{X\}$ and b is a limit. So in indescrete spaces limits may not be unique.

Lemma

Let (X, d) be a metric space. $A \subseteq X$ is a d-closed subset of X iff $\forall (x_n)_{n\in\mathbb{N}}$, with $x_n \in A$, $\forall n \in \mathbb{N}$ and $x_n \to x$ with respect to d, then $x \in A$.

Proof

Let $A \subseteq X$ be such that for all sequences, $(x_n)_{n\in\mathbb{N}}$, in A with $d - \lim x_n = x$ it holds that $x \in A$. Also assume that A is not d-closed. Thus,

$$\begin{aligned}
&A \text{ is not } d\text{-closed} \iff \\
&A' \text{ is not } d\text{-open} \iff \\
&\exists\, x \in A' : \forall \varepsilon > 0, \mathcal{O}_d(x, \varepsilon) \bigcap A \neq \varnothing
\end{aligned}$$

For every $n \in \mathbb{N}$ choose an x_n such that

$$x_n \in \mathcal{O}_d\left(x, \frac{1}{n+1}\right) \bigcap A$$

and hence construct a sequence in A.

Now consider a $\delta > 0$ such that for some n_δ

$$\delta > \frac{1}{n_\delta + 1} \iff \frac{1}{\delta} < n_\delta + 1 \iff n_\delta < \frac{1-\delta}{\delta} < +\infty$$

so that

$$\forall \delta > 0 \; \exists \; n_\delta \in \mathbb{N} : \forall n \geq n_\delta, \, x_n \in \mathcal{O}_d\left(x, \frac{1}{n+1}\right) \subseteq \mathcal{O}_d(x, \delta)$$

Thus $x_n \to x$, but $x \notin A$, which leads to a contradiction. So, for a set $A \subseteq X$ such that for all sequences, $(x_n)_{n\in\mathbb{N}}$, in A with $d - \lim x_n = x$ it holds that $x \in A$, it also has to hold that A is a d-closed subset of X.

For the converse suppose that some $A \in X$ is d-closed and that $\exists \; (x_n)_{n\in\mathbb{N}}$ in A with $x_n \to x$ and $x \notin A$. Then

$$\begin{aligned} &A \text{ is } d\text{-closed} \iff \\ &A' \text{ is } d\text{-open} \iff \\ &\exists \, \varepsilon > 0 : \mathcal{O}_d(x, \varepsilon) \bigcap A = \varnothing \end{aligned}$$

Since $x_n \to x$ almost every element of $(x_n)_{n\in\mathbb{N}}$ is in $\mathcal{O}_d(x, \varepsilon)$. But since this is a sequence in A and A and $\mathcal{O}_d(x, \varepsilon)$ are disjoint, this cannot hold. Contradiction! ■

So, we have found a way of characterizing d-closedness in metric spaces that is not in terms of d-openness (at least this is how it appears to be at a superficial level). One thing that we can do with this is "prove" that the empty set, $\varnothing$, is d-open and the carrier set, X, is d-closed. Consider a metric space, (X, d), and all d-convergent sequences in it. Inevitably, all of the limits of these sequences will lie in X. Thus, by the above lemma, X is a d-closed subset of itself. Consequently, $\varnothing$ is a d-open subset of X.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and f a function from X to Y, $f : X \to Y$. f is d_Y/d_X-continuous at $x \in X$ iff

$$\forall (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ with } x = d_X - \lim(x_n) \Rightarrow f(x) = d_Y - \lim(f(x_n))$$

Lemma

Let $f : X \to Y$ be some function with (X, d_X) and (Y, d_Y) metric spaces. Then the following statements are all equivalent to one another (i.e. when any one of them holds, all them concurrently hold) for any point $x \in X$:

1. f is d_Y/d_X-continuous at $x \in X$
2. $\forall \delta > 0, \exists \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$
3. If $A \in \tau_{d_Y}(f(x))$ then $\exists B \in \tau_{d_X}(x) : B \subseteq f^{-1}(A)$

Proof

We want to show that each of the above conditions is a *necessary* and *sufficient* condition for all of the others.

First, we show that 2. is a *necessary* and *sufficient* condition for 1.

Assume that 2. holds. Then $\forall \delta > 0, \exists\, \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$.

For $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $x_n, x \in X$ consider $(f(x_n))_{n\in\mathbb{N}}$. Then for some $\delta > 0$, choose a ε_δ that satisfies 2. Because $x_n \to x$

$$\forall n \geq n^*(\varepsilon_\delta),\, x_n \in \mathcal{O}_{d_X}(x, \varepsilon_\delta) \Rightarrow$$
$$\forall n \geq n^*(\varepsilon_\delta),\, f(x_n) \in f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right)$$

and because we assumed that $f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$

$$\forall n \geq n^*(\varepsilon_\delta),\; f(x_n) \in \mathcal{O}_{d_Y}(f(x), \delta)$$

and since δ is arbitrary $f(x_n) \to f(x)$. Because $(x_n)_{n\in\mathbb{N}}$ is arbitrary f is d_Y/d_X-continuous at $x \in X$.

So 2. is a *sufficient* condition for 1. at x.

Now, suppose that 1. holds (f is d_Y/d_X-continuous at $x \in X$), but for some $\delta > 0$ no ε_δ exists that satisfies 2. and $\forall \varepsilon > 0, f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x),\delta)$. This can be equivalently expressed as

$$\exists \delta > 0 : \forall \varepsilon > 0, f(\mathcal{O}_{d_X}(x,\varepsilon)) \bigcap \mathcal{O}'_{d_Y}(f(x),\delta) \neq \varnothing$$

This implies that

$$\exists \delta > 0 : \forall n \in \mathbb{N}, f(\mathcal{O}_{d_X}(x,\frac{1}{n+1})) \bigcap \mathcal{O}'_{d_Y}(f(x),\delta) \neq \varnothing$$

(since all $n \in \mathbb{N}$ give suitable ε). Consider the images of these sets through f^{-1} (which are also non-empty)*

$$f^{-1}\left(f\left(\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\right)\bigcap \mathcal{O}'_{d_Y}\left(f(x),\delta\right)\right) = \mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \neq \varnothing, \forall n \in \mathbb{N}$$

and a sequence, $(x_n)_{n\in\mathbb{N}}$, such that the n-th element of the sequence belongs to the n-th such set

$$\begin{aligned} &x_n \in \mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow \\ &x_n \in \mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right) \end{aligned}$$

which can be shown to imply a convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x. Thus, $x = d_X - \lim(x_n)$.

By the assumed d_Y/d_X-continuity of f at x, we get $f(x) = d_Y - \lim(f(x_n))$, but

$$\begin{aligned} &x_n \in \mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow \\ &x_n \in f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \iff \\ &f(x_n) \in f(f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))) = \mathcal{O}'_{d_Y}(f(x),\delta) \iff \\ &f(x_n) \notin \mathcal{O}_{d_Y}(f(x),\delta) \end{aligned}$$

which can be shown to make convergence of $(f(x_n))_{n\in\mathbb{N}}$ at $f(x) \in Y$ impossible. Hence we have a contradiction.

So 2. is a *necessary* condition for 1. at x.

Now, we show that 2. and 3. imply one another.

Let 3. hold.

*Even if f is not one-to-one, we can think of f^{-1} as a function $f^{-1} : Y \to P(X)$ that maps elements of Y to the subsets of X whose elements are mapped to the given value, i.e. $f^{-1}(y) = \{x \in X | f(x) = y\}$. Furthermore, we can think of f and f^{-1} as mappings between subsets of X and Y, not just single elements, which is how we are using them here. In this case, the reverse image we are taking is not empty, because f is used to produce the set we are passing to f^{-1}.

For $\delta > 0$ choose $A = \mathcal{O}_{d_Y}(f(x), \delta)$. By our assumption $\exists\ B$ in the neighbourhood system $\tau_{d_X}(x)$ such that $B \subseteq f^{-1}(A)$. Since $B \in \tau_{d_X}(x)$ there always exists a $\varepsilon > 0$ such that B is a subset of a d_X-open ball with center x and radius ε. All this implies that

$$\begin{aligned} \mathcal{O}_{d_X}(x, \varepsilon) &\subseteq B \subseteq f^{-1}(A) \Rightarrow \\ \mathcal{O}_{d_X}(x, \varepsilon) &\subseteq f^{-1}(\mathcal{O}_{d_Y}(f(x), \delta)) \Rightarrow \\ f(\mathcal{O}_{d_X}(x, \varepsilon)) &\subseteq f(f^{-1}(\mathcal{O}_{d_Y}(f(x), \delta))) \Rightarrow \\ f(\mathcal{O}_{d_X}(x, \varepsilon)) &\subseteq \mathcal{O}_{d_Y}(f(x), \delta) \end{aligned}$$

So 3. implies 2.

Now, let 2. hold.

Suppose that $\exists A \in \tau_{d_Y}(f(x))$ such that $\forall B \in \tau_{d_X}(x)$, B is not a subset of $f^{-1}(A)$, i.e.

$$B \bigcap \left(f^{-1}(A)\right)' \neq \varnothing \qquad \forall B \in \tau_{d_X}(x)$$

Because all d_X-open balls with center x belong to $\tau_{d_X}(x)$

$$\begin{aligned} \mathcal{O}_{d_X}(x, \varepsilon) \bigcap \left(f^{-1}(A)\right)' &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ \mathcal{O}_{d_X}(x, \varepsilon) \bigcap f^{-1}(A') &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon) \bigcap f^{-1}(A')\right) &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap f\left(f^{-1}(A')\right) &\neq \varnothing & \forall \varepsilon > 0 \Rightarrow \\ f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap A' &\neq \varnothing & \forall \varepsilon > 0 \end{aligned}$$

But $A \in \tau_{d_Y}(f(x))$ so there always exists $\delta > 0 : \mathcal{O}_{d_Y}(f(x), \delta) \subseteq A$. This implies that $(\mathcal{O}_{d_Y}(f(x), \delta))' \supseteq A'$ and thus

$$f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \bigcap (\mathcal{O}_{d_Y}(f(x), \delta))' \neq \varnothing \qquad \forall \varepsilon > 0$$

which is equivalent to $f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x), \delta)$ and contradicts 2.

So 2. implies 3. $\blacksquare$