Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
Prof: Stelios Arvanitis
TA: Dimitris Zaverdas*

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Problem Set 2

Open and Closed Balls, Boundness, and Total Boundness

Exercise 1

Show that open and closed balls can be defined for pseudo-metric spaces.

Let (X, d) be a pseudo-metric space (i.e d is a pseudo-metric on $X \neq \varnothing$). For $x \in X$ and $\varepsilon > 0$ we can try and define d-open balls with center x and radius ε as

$$\mathcal{O}_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

and d-closed balls with center x and radius ε as

$$\mathcal{O}_d[x, \varepsilon] = \{y \in X : d(x, y) \leq \varepsilon\}$$

To argue that $\mathcal{O}_d(x, \varepsilon)$ and $\mathcal{O}_d[x, \varepsilon]$ can be defined when d is a pseudo-metric, it suffices to show that they are non-empty sets for any x and ε. Notice the following, for d pseudo-metric on X

$$\forall x \in X, \varepsilon > 0 \; \exists \, y \in X : d(x, y) = 0 < \varepsilon$$

one of which is x itself, but there can also be other such $y \in X$ with $y \neq x$.
So $\exists \, y \in X : y \in \mathcal{O}_d(x, \varepsilon)$, so $\mathcal{O}_d(x, \varepsilon) \neq \varnothing$. Analogously, $\mathcal{O}_d[x, \varepsilon] \neq \varnothing$. So open and closed balls can be defined for pseudo-metric spaces.

Exercise 2

For the following (X, d) pairs, show that they constitute (pseudo-)metric spaces and define the unit open balls on them and visualize them:

1. $X = \mathbb{R}$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x, y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases} \quad , \forall x, y \in X$$

   with $c > 0$.

*Please report any typos, mistakes, or even suggestions at zaverdasd@aueb.gr.

It can be easily shown that d is a metric on X (discrete metric). Now, the unit open ball in (X, d) is

$$\begin{aligned}\mathcal{O}_d(0,1) &= \{x \in X : d(x,0) < 1\} \\ &= \{x \in X : d(x,0) = \begin{cases} 0, & x = 0 \\ c, & x \neq 0 \end{cases} < 1\}\end{aligned}$$

and we need to examine cases for the values c may take.

If $0 < c < 1$, then any element in X has a distance from 0 that is smaller than 1. So $\mathcal{O}_d(0,1) = \mathbb{R}$, and if one were to visualize it on a graph of the real line it would cover the entire graph.

If $c \geq 1$, then all elements in X except for 0 have a distance from 0 that is greater than or equal to 1. So $\mathcal{O}_d(0,1) = \{0\}$, and it were to be visualized on a real line it would be a single point at 0.

2. $X = \mathbb{R}^2$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x,y) = \sqrt{(x-y)'A(x-y)},\ \forall x, y \in X$$

with $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

It can be shown that properties i, iii, and iv hold for d on X. Observe that for any arbitrary $x, y \in X$

$$\begin{aligned} d(x,y) &= \sqrt{(x-y)'A(x-y)} \\ &= \sqrt{\begin{bmatrix} x_1 - y_1 & x_2 - y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}} \\ &= \sqrt{\begin{bmatrix} 1 \cdot (x_1 - y_1) + 0 \cdot (x_2 - y_2) & 0 \cdot (x_1 - y_1) + 0 \cdot (x_2 - y_2) \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}} \\ &= \sqrt{\begin{bmatrix} x_1 - y_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \end{bmatrix}} \\ &= \sqrt{(x_1 - y_1)^2} \\ &= |x_1 - y_1| \end{aligned}$$

which means that $d(x,x) = 0$ always, but also for $x \neq y : x_1 = y_1 \Rightarrow d(x,y) = 0$. So d is a pseudo-metric on X.

If $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, the unit open ball in this space is

$$\begin{aligned} \mathcal{O}_d(\vec{0}, 1) &= \{y \in \mathbb{R}^2 : d_A(\vec{0}, y) < 1\} \\ &= \{y \in \mathbb{R}^2 : |0 - y_1| < 1\} \\ &= \{y \in \mathbb{R}^2 : |y_1| < 1\} \\ &= \{y \in \mathbb{R}^2 : -1 < y_1 < 1, y_2 \in \mathbb{R}\} \\ &= (-1, 1) \times \mathbb{R} \end{aligned}$$

If it were to be visualized on a 2-D graph, it would cover the entire area between the vertical lines at $x_1 = -1$ and $x_1 = 1$ (but not the lines themselves).

3. $X = \mathbb{R}^3$ and $d : X \times X \to \mathbb{R}$, such that

$$d(x, y) = \max\left\{|x_1 - y_1|, \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2}\right\}, \forall x, y \in X$$

It can be shown that d is a metric on X. Observe that for any arbitrary $x, y \in X$

$$\begin{aligned} d(x, y) &= \max\left\{|x_1 - y_1|, \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2}\right\} \\ &= \begin{cases} |x_1 - y_1|, & |x_1 - y_1| \geq \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2} \\ \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2}, & |x_1 - y_1| < \sqrt{(x_2 - y_2)^2 + (x_3 - y_3)^2} \end{cases} \end{aligned}$$

If $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, the unit open ball in this space is

$$\begin{aligned} \mathcal{O}_d(\vec{0}, 1) &= \{x \in \mathbb{R}^3 : d(x, \vec{0}) < 1\} \\ &= \left\{x \in \mathbb{R}^3 : \max\left\{|x_1 - 0|, \sqrt{(x_2 - 0)^2 + (x_3 - 0)^2}\right\} < 1\right\} \\ &= \left\{x \in \mathbb{R}^3 : \begin{cases} |x_1|, & \sqrt{x_2^2 + x_3^2} \leq |x_1| \\ \sqrt{x_2^2 + x_3^2}, & |x_1| < \sqrt{x_2^2 + x_3^2} \end{cases} < 1\right\} \\ &= \left\{x \in \mathbb{R}^3 : |x_1| < 1 \text{ and } \sqrt{x_2^2 + x_3^2} < 1\right\} \\ &= (-1, 1) \times \{(\kappa, \lambda) \in \mathbb{R}^2 : \kappa < \sqrt{1 - \lambda^2}\} \end{aligned}$$

If it were to be visualized on a 3-D graph, it would look like the interior (without the borders) of a cylinder centered at $(0, 0, 0)$ with radius 1 and height 2. The circular faces would be perpendicular to the first axis at $x_1 = -1$ and $x_1 = 1$.

Exercise 3

Let (X, d) be a metric space and for some $x, y \in X$ and $\varepsilon > 0$ let $y \in \mathcal{O}_d(x, \varepsilon)$. Show that $\exists\, \delta > 0 : \mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$.

Because $y \in \mathcal{O}_d(x, \varepsilon) \iff d(x, y) < \varepsilon \iff 0 < \varepsilon - d(x, y)$. Choose $\delta := \varepsilon - d(x, y) > 0$.
Then $\mathcal{O}_d(y, \delta) = \{z \in X : d(y, z) < \delta\}$. So

$$\begin{aligned} z &\in \mathcal{O}_d(y, \delta) \\ d(y, z) &< \delta \\ d(x, y) + d(y, z) &< \delta + d(x, y) \\ d(x, z) \leq d(x, y) + d(y, z) &< \delta + d(x, y) = \varepsilon \\ d(x, z) &< \varepsilon \\ z &\in \mathcal{O}_d(x, \varepsilon) \end{aligned}$$

i.e. $z \in \mathcal{O}_d(y, \delta) \Rightarrow z \in \mathcal{O}_d(x, \varepsilon)$, which says that every element of $\mathcal{O}_d(y, \delta)$ also belongs to $\mathcal{O}_d(x, \varepsilon)$, so $\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$ for this choice of δ.

Exercise 4

Let (X, d) be a metric space and $Y \subseteq X$. Define $d' : Y \times Y \to \mathbb{R}$ such that $d' = d|_{Y \times Y}$. Then (Y, d') is a metric subspace of (X, d). Show that $\mathcal{O}_{d'}(x, \varepsilon) = \mathcal{O}_d(x, \varepsilon) \cap Y$ and $\mathcal{O}_{d'}[x, \varepsilon] = \mathcal{O}_d[x, \varepsilon] \cap Y$.

Observe that arithmetically $d'(x, y) = d(x, y)$, $\forall x, y \in Y$ and that

$$Y \subseteq X \iff (z \in Y \Rightarrow z \in X)$$

Furthermore

$$z \in \mathcal{O}_d(x, \varepsilon) \iff z \in X \text{ and } d(z, x) < \varepsilon$$

Thus

$$\begin{aligned} z \in \mathcal{O}'_d(y, \varepsilon) &\iff z \in Y \text{ and } d'(z, x) < \varepsilon \\ &\Rightarrow z \in X \text{ and } d(z, x) < \varepsilon \\ &\iff z \in \mathcal{O}_d(x, \varepsilon) \end{aligned}$$

So $z \in \mathcal{O}_{d'}(x, \varepsilon) \iff (z \in Y \text{ and } z \in \mathcal{O}_d(x, \varepsilon))$ which means that $\mathcal{O}_{d'}(x, \varepsilon) = \mathcal{O}_d(x, \varepsilon) \cap Y$.
Similarly, $\mathcal{O}_{d'}[x, \varepsilon] = \mathcal{O}_d[x, \varepsilon] \cap Y$.

Exercise 5

Is $(0, 1)$ a bounded set?

Boundness is not a topological notion. A set can only be a bounded subset of some other set with respect to a specified metric function. Furthermore, a set may be bounded or not bounded depending on the chosen metric.

Exercise 6

Let $d : \mathbb{R}_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ such that $d(x, y) = |ln(x) - ln(y)|$, $\forall x, y \in \mathbb{R}_{++}$ be a metric on $\mathbb{R}_{++}$. Is $(0, 1)$ a d-bounded subset of $\mathbb{R}_{++}$?

A set is a bounded subset in a metric space (i.e. with respect to a specific metric) if there exists an open (closed) ball in the space that can cover it.

Let $x \in \mathbb{R}_{++}$, $y \in (0, 1)$, and $\varepsilon > 0$. Define the open ball $\mathcal{O}_d(x, \varepsilon)$. Choose a y in $(0, 1)$ such that

$$y < x \iff ln(y) < ln(x) \iff |ln(x) - ln(y)| = ln(x) - ln(y) \iff d(x, y) = ln(x) - ln(y)$$

For y to belong to $\mathcal{O}_d(x, \varepsilon)$ it must hold that

$$d(x, y) < \varepsilon \overset{y<x}{\iff} ln(x) - ln(y) < \varepsilon \iff -ln(y) < \varepsilon - ln(x) \iff ln(y) > ln(x) - \varepsilon \iff y > e^{ln(x)-\varepsilon} > 0$$

Thus, for y to belong to $\mathcal{O}_d(x, \varepsilon)$, it must be bounded strictly away from 0. So for any arbitrary d-open ball in $\mathbb{R}_{++}$, $\mathcal{O}_d(x, \varepsilon)$, there always exists a $0 < y' < e^{ln(x)-\varepsilon}$ that does not belong to it. So $(0, 1)$ cannot be a d-bounded subset of $\mathbb{R}_{++}$.

Exercise 7

Let (Y, d) be a metric space and $X \neq \varnothing$. For $d_{sup} : \mathcal{B}(X, Y) \times \mathcal{B}(X, Y) \to \mathbb{R}$ such that

$$d_{sup}(f, g) = \sup_{x \in X} d(f(x), g(x)), \ \forall f, g \in \mathcal{B}(X, Y)$$

show that:

1. $(\mathcal{B}(X, Y), d_{sup})$ is a metric space.

   For $(\mathcal{B}(X, Y), d_{sup})$ to be a metric space, $\mathcal{B}(X, Y)$ needs to be a non-empty set and d_{sup} needs to satisfy properties i-iv on $\mathcal{B}(X, Y)$.

   First, notice that X and Y are non-empty, so we can define some "constant" function $f_c : X \to Y$ such that $f_c(x) = y_c$, $\forall x \in X$ for some $y_c \in Y$. Furthermore, observe that $f_c(X) = \{y_c\} \subseteq Y$, i.e. the image of X through f_c is a subset of Y (naturally) and it is also a singleton set (it has only one element). Thus $f(X)$ is a bounded subset of Y. That is, we have found at least one example of a bounded function from X to Y. So $\mathcal{B}(X, Y)$ is non-empty.

   Secondly, since (Y, d) is a metric space, the properties i-iv hold for d on Y. So

   i) $d_{sup}(f, g) = \sup_{x \in X} d(f(x), g(x)) \geq 0$, $\forall f, g \in \mathcal{B}(X, Y)$

   ii) $d(x, y) = 0 \iff x = y, \forall x, y \in X$, so for some arbitrary $f, g \in \mathcal{B}(X, Y)$

$$d(f(x), g(x)) = 0 \iff f(x) = g(x)$$
$$\sup_{x \in X} d(f(x), g(x)) = 0 \iff f(x) = g(x), \ \forall x \in X$$
$$\sup_{x \in X} d(f(x), g(x)) = 0 \iff f = g$$
$$d_{sup}(f, g) = 0 \iff f = g$$

And since f and g were chosen arbitrarily, $d_{sup}(f,g) = 0 \iff f = g$, $\forall f, g \in \mathcal{B}(X,Y)$.

iii) $d_{sup}(f,g) = \sup_{x\in X} d(f(x), g(x)) = \sup_{x\in X} d(g(x), f(x)) = d_{sup}(g,f)$, $\forall f, g \in \mathcal{B}(X,Y)$

iv) $d_{sup}(f,g) = \sup_{x\in X} d(f(x), g(x)) \leq \sup_{x\in X} \Big(d(f(x), h(x)) + d(h(x), g(x))\Big)$
$\leq \sup_{x\in X} d(f(x), h(x)) + \sup_{x\in X} d(h(x), g(x)) = d_{sup}(f,h) + d_{sup}(h,g)$, $\forall f, g, h \in \mathcal{B}(X,Y)$

So $(\mathcal{B}(X,Y), d_{sup})$ is a metric space.

2. If (Y,d) is bounded, then $(\mathcal{B}(X,Y), d_{sup})$ is also bounded.

   (Y,d) being bounded means that there exist an $x' \in Y$ and an $\varepsilon > 0$ such that $Y \subseteq \mathcal{O}_d(x', \varepsilon)$, which by definition means that $d(x', y) < \varepsilon$, $\forall y \in Y$. Define the function $f_c \in \mathcal{B}(X,Y)$ such that $f_c(x) = x', \forall x \in X$. Also don't forget that

$$f \in \mathcal{B}(X,Y) \Rightarrow f : X \to Y \Rightarrow f(x) \in Y,\ \forall x \in X$$

   So $\forall f \in \mathcal{B}(X,Y)$

$$d_{sup}(f_c, f) = \sup_{x\in X} d(f_c(x), f(x)) = \sup_{x\in X} d(x', f(x)) \leq \sup_{x\in X} \varepsilon = \varepsilon$$

   So $d_{sup}(f_c, f) < \varepsilon$, $\forall f \in \mathcal{B}(X,Y) \iff f \in \mathcal{O}_{d_{sup}}(f_c, \varepsilon)$, $\forall f \in \mathcal{B}(X,Y)$ and $\mathcal{B}(X,Y) \subseteq \mathcal{O}_{d_{sup}}(f_c, \varepsilon)$ so $\mathcal{B}(X,Y)$ is d_{sup}-bounded.

Exercise 8

Let $X \subseteq \mathbb{R}^N$ with $N \in \mathbb{N}^*$ and $d : X \times X \to \mathbb{R}$ such that $d(x,y) = \left(\sum_{i=1}^{N} |x_i - y_i|^2\right)^{\frac{1}{2}}$, $\forall x, y \in X$ be the Euclidean metric on X. Show that d-boundeness in X is sufficient for d-total boundeness in X.
(Hint: Consider a d-bounded set in X and show that the ball that covers it is d-totally bounded.)

For A to be d-totally bounded subset of X, there must exist **for any** $\varepsilon > 0$ a **finite** number of d-open (d-closed) balls in X that collectively include every element of A. This is called a finite cover of A.
Let A be a d-bounded subset of X. Then $\exists\, x_0 \in \mathbb{R}^N$, $\delta > 0 : A \subseteq \mathcal{O}_d[x_0, \delta]$.
We will prove that $\mathcal{O}_d[x_0, \delta]$ is d-totally bounded, thus A is also d-totally bounded as a subset of $\mathcal{O}_d[x_0, \delta]$.
For all $\varepsilon \geq \delta > 0$ all of $\mathcal{O}_d[x_0, \delta]$ can be covered by one d-closed ball, $\mathcal{O}_d[x_0, \varepsilon]$.
For any $0 < \varepsilon < \delta$. Notice that every dimension of $\mathcal{O}_d[x_0, \delta]$ is a subset of $[x_{0i} - \delta, x_{0i} + \delta]$, where x_{0i} is the i-th element of x_0, i.e.

$$\mathcal{O}_d[x_0, \delta] \subseteq [x_{01} - \delta, x_{01} + \delta] \times [x_{02} - \delta, x_{02} + \delta] \times \ldots \times [x_{0N} - \delta, x_{0N} + \delta]$$

Let $m \in \mathbb{N}^*$ be such that $m > \dfrac{2N\delta}{\varepsilon}$, then each of those supersets can be divided into exactly m subintervals like so

$$\begin{aligned}[x_{0i} - \delta, x_{0i} + \delta] = &\left[x_{0i} - \delta, x_{0i} - \delta + \frac{2\delta}{m}\right] \bigcup \\ &\left[x_{0i} - \delta + \frac{2\delta}{m}, x_{0i} - \delta + 2\frac{2\delta}{m}\right] \bigcup \\ &\ldots \\ &\left[x_{0i} + \delta - 2\frac{2\delta}{m}, x_{0i} + \delta - \frac{2\delta}{m}\right] \bigcup \\ &\left[x_{0i} + \delta - \frac{2\delta}{m}, x_{0i} + \delta\right]\end{aligned}$$

for all $i \in \{1, 2, ..., N\}$.

By restricting each dimension to one of those subintervals we can construct up to m^N distinct subsets of X (because we have m choices for each of the N dimensions). Those constructs are analogous to N-dimensional cubes with centers x_j such that

$$x_{ji} = \frac{(x_{0i} + s_{ji}) + \left(x_{0i} + s_{ji} + \frac{2\delta}{m}\right)}{2} = x_{0i} + s_{ji} + \frac{\delta}{m}$$

with $s_{ji} \in \left\{-\delta, -\delta + \frac{2\delta}{m}, ..., \delta - 2\frac{2\delta}{m}, \delta - \frac{2\delta}{m}\right\}$ the appropriate step to give the selected interval for the i-th dimension of x_j.

It can be shown that between all elements of those such cubes their "corner" elements have the maximum d-distance from their centers x_j and that is equal to $\frac{\delta\sqrt{N}}{m}$ (proof at the end of these notes). Furthermore, observe that

$$0 < \frac{\delta\sqrt{N}}{m} \leq \frac{\delta N}{m} \leq \frac{2\delta N}{m} < \varepsilon$$

So each of those cube sets can be "inscribed" into a d-closed ball, $\mathcal{O}_d[x_j, \varepsilon]$. Remember that there are exactly m^N such cube sets, i.e. there is a finite number of them and they all collectively cover $\mathcal{O}_d[x_0, \delta]$. Thus the finitely many balls that cover the collection of the cube sets also cover $\mathcal{O}_d[x_0, \delta]$. Remember that $\varepsilon > 0$ was arbitrarily chosen, so there exists a finite cover of $\mathcal{O}_d[x_0, \delta]$ for all ε. So $\mathcal{O}_d[x_0, \delta]$ is a d-totally bounded subset of X and because $A \subseteq \mathcal{O}_d[x_0, \delta]$ it is also d-totally bounded.

So d-boundness in $X \subseteq \mathbb{R}^N$ is sufficient for d-total boundness in X.

We can generalize this result and say that boundness in finite Euclidean spaces is equivalent to total boundness.

It is worth noticing that $m^N = \left(\frac{2N\delta}{\varepsilon}\right)^N$ gives the *upper bound of the covering number* of the chosen d-bounded set, A, and that it is decreasing in ε and increasing in N.

Exercise 9

Let $X = \{(x_n)_{n\in\mathbb{N}^*} : x_n \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 < +\infty\}$ (i.e. X is the set of square summable real sequences) and $d : X \times X \to \mathbb{R}$ such that $d(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{\frac{1}{2}}$, $\forall x, y \in X$ be a metric on X. Show that d-boundeness in X is not sufficient for d-total boundeness in X.

(Hint: Consider the sequence $\mathbf{0} = \{0\}_{n\in\mathbb{N}^*} \in X$ and the d-closed unit ball centered at it. Use Riesz's Lemma and the Pigeonhole Principle.)

We can get an intuitive understanding of why in an infini-dimensional space (Hilbert space) there can be no **finite** cover of any of its bounded subsets by studying the behaviour as $N \to \infty$ of the upper bound of the covering number of such a set as it is given by the solution of exercise 8 for $\varepsilon < \delta$.

$$\lim_{N\to\infty} m^N = \lim_{N\to\infty} \left(\frac{2N\delta}{\varepsilon}\right)^N = \lim_{N\to\infty} e^{N\cdot ln\left(\frac{2N\delta}{\varepsilon}\right)} = \lim_{N\to\infty} e^{N\cdot ln(N) + N\cdot ln(2) + N\cdot(ln(\delta) - ln(\varepsilon))} \overset{\varepsilon\leq\delta}{=} \infty$$

In a way, this is total boundness getting "lost in the limit".

However, the above is not a complete proof. We will prove that d-boundness in X does not necessarily imply d-total boundness by proving that we can construct a counter-example (and we will briefly see a counter-example after the proof).

Denote by $\mathbf{0} = \{0\}_{n\in\mathbb{N}^*}$ the sequence with elements all equal to zero and consider the d-closed unit ball, $\mathcal{O}_d[\mathbf{0}, 1]$, which has an infinite number of elements. Since $\mathcal{O}_d[\mathbf{0}, 1]$ is a d-closed ball, it is d-bounded (by itself).
For any $\varepsilon \geq 1$ obviously $\mathcal{O}_d[\mathbf{0}, 1]$ can be covered by one d-closed ball, $\mathcal{O}_d[\mathbf{0}, \varepsilon]$.
For any $0 < \varepsilon < 1$ we will prove that we can construct a counter-example (and one such counter-example is given after the proof). We will use Riesz's Lemma to progressively construct an infinite set of sequences in $\mathcal{O}_d[\mathbf{0}, 1]$ such that for each sequence, say x_n, it is $d(x_n, x_m) > \varepsilon$, $\forall m < n$. For a proof of Riesz's Lemma see Ch. 12 p. 221 of O'Searcoid's textbook.
Choose a sequence $x_1 \in \mathcal{O}_d[\mathbf{0}, 1]$ and without loss of generality, let $d(\mathbf{0}, x_1) = 1$. Denote by X_1 the set of all elements of X that fall on the line defined by $\mathbf{0}$ and x_1. (X_1, d) is a linear subspace of (X, d) because all linear combinations of elements of X_1 also belong in X_1. $X_1 \not\subset \mathcal{O}_d[\mathbf{0}, 1]$, but since (X_1, d) is linear, by Riesz's Lemma

$$\exists\, x_2 \in X :\ d(\mathbf{0}, x_2) = 1 | d(x_2, x) > \varepsilon,\ \forall x \in X_1$$

for the chosen ε. This x_2 is not on the line given by X_1, but is an element of $\mathcal{O}_d[\mathbf{0}, 1]$ (because by Riesz's Lemma it is of unit norm). Furthermore, it is such that $d(x_2, x_1) > \varepsilon$.
Now denote by X_2 the set of all elements of X that fall on the 2D surface defined by $\mathbf{0}$, x_1, and x_2. (X_2, d) is also a linear subspace of (X, d), and by Riesz's Lemma

$$\exists\, x_3 \in X :\ d(\mathbf{0}, x_3) = 1 | d(x_3, x) > \varepsilon,\ \forall x \in X_2$$

for the chosen ε, and this x_3 is not on the 2D surface given by X_2. Now $x_3 \in \mathcal{O}_d[\mathbf{0}, 1]$ and $d(x_3, x_1) > \varepsilon$ and $d(x_3, x_2) > \varepsilon$.
We can then denote by X_3 the 3D space defined by $\mathbf{0}$, x_1, x_2, and x_3, which again defines a linear subspace of (X, d), and find a x_4 whose d-distance from all the previous sequences is greater than ε.
And so on...
Generally, in the n-th step we can denote by (X_n, d) some linear subspace of (X, d) that is defined by all the previous sequences and $\mathbf{0}$ and find a sequence $x_{n+1} \in \mathcal{O}_d[\mathbf{0}, 1]$ such that its distance from all of them is greater than ε. Since X has an infinite number of dimensions, we can find an infinite number of such sequences in X.
So, for $0 < \varepsilon < 1$, consider the set of all these sequences

$$A = \{x_n \in \mathcal{O}_d[\mathbf{0}, 1] :\ n \in \mathbb{N}^*,\ d(x_m, x_n) > \varepsilon\ \forall m \neq n \in \mathbb{N}\}$$

Suppose that all other elements of $\mathcal{O}_d[\mathbf{0}, 1]$ can collectively be covered by a finite number of balls. For $\mathcal{O}_d[\mathbf{0}, 1]$ to be d-totally bounded, A must also be covered by a finite number of balls.
Let n be the number of elements in A and $0 < m < \infty$ a finite number of balls of radius ε with which we wish to cover A. By the Pigeonhole Principle at least one ball must include at least k elements, where it holds for n, m, and k that

$$n = km + 1 \iff k = \frac{n-1}{m}$$

But because $n = \infty \Rightarrow k = \infty > 1$ (i.e. at least one ball must contain an infinte number of elements). However, all elements of A are such that the d-distance between them is greater than ε which is the radius of the balls we are using. So no ball can cover more than one element. Contradiction!

Thus A is not d-totally bounded and $\mathcal{O}_d[\mathbf{0}, 1]$ is not d-totally bounded and d-boundeness in X is not sufficient for d-total boundness in X.

An example of an infinite set of sequences in $\mathcal{O}_d[\mathbf{0}, 1]$ that are "bounded away" from each other is the following set of "basis sequences" in X

$$\{b_n : b_{ni} = \begin{cases} 0 & i \neq n \\ 1 & i = n \end{cases}, \forall n \in \mathbb{N}^*\}$$

Additionally, ponder on why the approach taken for this proof would fail to show that d-bounded subsets of X are not d-totally bounded when $N < \infty$ (i.e. in Euclidean spaces).

Exercise 10

Let $X = \{f : [0, 1] \to \mathbb{R} | \int_0^1 f^2(x)dx < +\infty\}$ be the set of square integrable functions from $[0, 1]$ to $\mathbb{R}$. Consider the metric function $d(f, g) \coloneqq \left(\int_0^1 (f(x) - g(x))^2 dx\right)^{\frac{1}{2}}$ on X. If $\mathbf{0} : [0, 1] \to \mathbb{R}$ is a function in X such that $\mathbf{0}(x) \coloneqq 0 \ \forall x \in [0, 1]$, consider $\mathcal{O}_d[\mathbf{0}, 1]$ and show that it is not d-totally bounded.

Sketch of proof...

Take the general intuition behind the proof in Exercise 9, which involves a countably infinite carrier set. Loosely speaking, because there is an infinite number of dimensions in the set that is being examined, we can always "escape" away from any proposed finite set of covers by moving towards other dimensions. We can work similarly for an uncountably infinite carrier set and see that a counter-example can be constructed.

Exercise 11

Let (X_i, d_i) be metric spaces $\forall i \in \mathcal{I}$ with $\mathcal{I}$ a finite index set. For the cartesian product $X \coloneqq \prod_{i \in \mathcal{I}} X_i$ there can be defined the following structured sets (X, d_Π) with $d_\Pi \in \{d_{\Pi_{max}}, d_{\Pi_I}, d_{\Pi_{||}}\}$ and d_Π are defined as

$$d_{\Pi_{max}} = \max_{i \in \mathcal{I}} d_i$$
$$d_{\Pi_I} = \left(\sum_{i \in \mathcal{I}} d_i^2\right)^{\frac{1}{2}}$$
$$d_{\Pi_{||}} = \sum_{i \in \mathcal{I}} d_i$$

and are appropriate metric functions on X. Let $A_i \subseteq X_i, \forall i \in \mathcal{I}$ and $A \coloneqq \prod_{i \in \mathcal{I}} A_i$, which implies that $A \subseteq X$. Show that A is a d_Π-totally bounded subset of X iff A_i are d_i-totally bounded subsets of $X_i \ \forall i \in \mathcal{I}$, for each of the three d_Π defined above.

Because $d_{\Pi_{max}} \leq d_{\Pi_I} \leq d_{\Pi_{||}} \leq n d_{\Pi_{max}}$ it suffices to show that

$$A \text{ is } d_{\Pi_{max}}\text{-totally bounded subset of } X \iff A_i \text{ is } d_i\text{-totally bounded subset of } X_i \ \forall i \in \mathcal{I}$$

where $n \in \mathbb{N}^*$ is the number of elements in $\mathcal{I}$, and the total boundeness property for a specific set (say A) is inherited by d_{Π_I} and $d_{\Pi_{||}}$ from $d_{\Pi_{max}}$, and by $d_{\Pi_{max}}$ from the other two.

A compact illustration of the proof is the following (statements above the $\iff$ sign describe how to move forward, while statements below show the way back)

$$\forall i \in \mathcal{I},\ A_i \text{ is } d_i\text{-totally bounded subset of } X_i \iff$$

$$\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists C_{A_i,\varepsilon_i} := \{\mathcal{O}_{d_i}(x_{ij}, \varepsilon_i), x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite}\} : A_i \subseteq \bigcup_{j \in \mathcal{I}_i} \mathcal{O}_{d_i}(x_{ij}, \varepsilon_i) \iff$$

$$\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, y_i \in \mathcal{O}_{d_i}(x_{ij}, \varepsilon_i) \iff$$

$$\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, d_i(x_{ij}, y_i) < \varepsilon_i \underset{\text{Choose } \varepsilon_i = \varepsilon}{\iff}$$

$$\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, \max_{i \in \mathcal{I}} d_i(x_{ij}, y_i) < \max_{i \in \mathcal{I}} \varepsilon_i \overset{\varepsilon := \max_{i \in \mathcal{I}} \varepsilon_i, \overline{\mathcal{I}} = \bigcup_{i \in \mathcal{I}} \mathcal{I}_i}{\underset{\text{Choose } \mathcal{I}_i = \overline{\mathcal{I}}}{\iff}}$$

$$\forall \varepsilon > 0\ \exists x_j \in X, j \in \overline{\mathcal{I}} \text{ finite} : \forall y \in A, d_{\Pi_{max}}(x_j, y) < \varepsilon \iff$$

$$\forall \varepsilon > 0\ \exists x_j \in X, j \in \overline{\mathcal{I}} \text{ finite} : \forall y \in A, y \in \mathcal{O}_{d_{\Pi_{max}}}(x_j, \varepsilon) \iff$$

$$\forall \varepsilon > 0\ \exists C_{A,\varepsilon} := \{\mathcal{O}_{d_{\Pi_{max}}}(x_j, \varepsilon), x_j \in X, j \in \overline{\mathcal{I}} \text{ finite}\} : A \subseteq \bigcup_{j \in \overline{\mathcal{I}}} \mathcal{O}_{d_{\Pi_{max}}}(x_j, \varepsilon) \iff$$

$$A \text{ is } d_{\Pi_{max}}\text{-totally bounded subset of } X$$

More verbosely, if A_i are d_i-totally bounded subsets of X_i for all $i \in \mathcal{I}$, then there exist $\forall i$ finite d_i-open covers for any $\varepsilon_i > 0$ (we choose a different ε_i for each A_i).
That means that each element, y_i, of each A_i belongs to some d_i-open ball with radius ε_i and the number of these balls (as well as their centres, x_{ij}) is finite for all i.
We construct elements of X and A using the above x_{ij} and y_i. If we consider the $d_{\Pi_{max}}$ metric on X, we can see that the distance of each $y = (y_n)_{n \in \mathcal{I}}$ in A from each $x_j = (x_{nj})_{n \in \mathcal{I}}$ in X, given by $d_{\Pi_{max}}$, is equal to the greatest distance between their elements, given by the corresponding d_i, i.e. $\forall x_j, y$

$$d_{\Pi_{max}}(x_j, y) = \max_{i \in \mathcal{I}} d_i(x_{ij}, y_i)$$

So we can construct $d_{\Pi_{max}}$-open balls using the x_j-s as centres and setting $\varepsilon := \max_{i \in \mathcal{I}}$ as their radii and cover all of A with them. Their number is finite.
Thus, we have constructed a finite $d_{\Pi_{max}}$-open cover of A **for all** $\varepsilon > 0$ using the fact that A_i are d_i-totally bounded subsets of X_i for all $i \in \mathcal{I}$. So A is a $d_{\Pi_{max}}$-totally bounded subset of X.

Conversely, if A is a $d_{\Pi_{max}}$- totally bounded subset of X, then for all $\varepsilon > 0$ there exists a finite cover of $d_{\Pi_{max}}$-open balls with radius ε, such that $\forall y \in A$, y belongs to one of these (finitely many) balls.
By definition of $d_{\Pi_{max}}$, the above means that each element of y, y_i, will belong to a d_i-open ball of radius ε. For all i the union of these balls covers each A_i and their number is the same as the number of balls used to cover A, which is finite.
Thus we have constructed finite d_i-open covers of A_i and A_i are d_i-totally bounded subsets of X_i for all i.

A few remarks:

- For a subset in a metric space to be totally bounded, a finite cover must exist **for all radii**. Make sure you see that this is the case here.

- Pay attention to the possibility that the index sets of each cover for the various i may not include the same indices. Thus, when constructing the index set for the cartesian product, we use their union ($\overline{\mathcal{I}} = \bigcup_{i \in \mathcal{I}} \mathcal{I}_i$). This means that we may need to use some x_{ij} that are not necessary to cover A_i, but are needed to construct the x_j that define the balls that cover A.

  E.g. $\mathcal{I} = \{1, 2\}$ and for some $\varepsilon_1, \varepsilon_2 > 0$ the index sets of the finite covers of A_1 and A_2 are $\mathcal{I}_1 = \{1, 2\}$ and $\mathcal{I}_2 = \{1, 2, 3\}$. Then we need $\overline{\mathcal{I}} = \{1, 2, 3\}$ to construct the cover of $A = A_1 \times A_2$ of radius $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$.

- The minimum effective size of a finite cover's index set depends on the size of the radius (and will typically converge to infinity as a radius approaches zero). But for all strictly positive radii, these index sets are finite.

Exercise 12

Let d_1 and d_2 be both metrics on a non empty set X such that $d_1 \leq cd_2$ with $c > 0$. Show that:

1. for $x \in X$ and $0 < \varepsilon$ then $\mathcal{O}_{d_2}(x, \varepsilon) \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon)$.

   Let $y \in \mathcal{O}_{d_2}(x, \varepsilon)$, then

$$\begin{aligned} d_2(x, y) &< \varepsilon \\ c \cdot d_2(x, y) &< c \cdot \varepsilon \\ d_1(x, y) \leq c \cdot d_2(x, y) &< c \cdot \varepsilon \\ d_1(x, y) &< c \cdot \varepsilon \end{aligned}$$

   which means that $y \in \mathcal{O}_{d_1}(x, c \cdot \varepsilon)$. So $\mathcal{O}_{d_2}(x, \varepsilon) \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon)$.

2. if $A \subseteq X$ is d_2-bounded, then it is d_1-bounded.

   Since A is d_2-bounded, $\exists\, x \in X, \varepsilon > 0 : A \subseteq \mathcal{O}_{d_2}(x, \varepsilon)$. Set $\delta \coloneqq c \cdot \varepsilon \Rightarrow \delta > 0$. So

$$\begin{aligned} &A \subseteq \mathcal{O}_{d_2}(x, \varepsilon) \\ &A \subseteq \mathcal{O}_{d_2}(x, \varepsilon) \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon) \\ &A \subseteq \mathcal{O}_{d_1}(x, c \cdot \varepsilon) \\ &A \subseteq \mathcal{O}_{d_1}(x, \delta) \end{aligned}$$

   So there $\exists\, x \in X, \delta > 0 : A \subseteq \mathcal{O}_{d_1}(x, \delta)$ and A is d_1-bounded.

3. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1-bounded iff it is d_2-bounded.

   Let A be d_2-bounded. Since $\exists\, c > 0 : d_1 \leq c \cdot d_2$, it follows that A is d_1-bounded. For this reason, if A is not d_1-bounded, it cannot be d_2-bounded.

   Let A be d_1-bounded. Since $\exists\, c' > 0 : d_2 \leq \dfrac{1}{c} \cdot d_2$, it follows that A is d_2-bounded. For this reason, if A is not d_2-bounded, it cannot be d_1-bounded.

   So if $\exists\, c > 0, c' > 0 : c'd_2 \leq d_1 \leq cd_2$, A is d_1-bounded iff it is d_2-bounded.

4. if $A \subseteq X$ is d_2-totally bounded, then it is d_1-totally bounded.

Since A is d_2-totally bounded, there must exist for any given radius a finite collection of d-open (d-closed) balls in X that collectively cover A (i.e. include all of its elements). That is called a finite cover of A of said radius.

Let $\varepsilon > 0$ be an arbitrary positive real number, then

$$\exists\, C_{A,\frac{\varepsilon}{c}} := \left\{\mathcal{O}_{d_2}\left(x_i, \frac{\varepsilon}{c}\right), x_i \in X, i \in \mathcal{I} \text{ finite}\right\} : A \subseteq \bigcup_{i \in \mathcal{I}} \mathcal{O}_{d_2}\left(x_i, \frac{\varepsilon}{c}\right)$$

where $C_{A,\frac{\varepsilon}{c}}$ is one such finite cover of A of radius $\frac{\varepsilon}{c}$ (and it is not necessarily unique).

Since $d_1 \leq c \cdot d_2 \Rightarrow \mathcal{O}_{d_2}\left(x_i, \frac{\varepsilon}{c}\right) \subseteq \mathcal{O}_{d_1}(x_i, \varepsilon), \forall i \in \mathcal{I}$. So this collection of d_1-open balls of radius ε covers the corresponding collection of d_2-open balls of radius $\frac{\varepsilon}{c}$, which also covers A. Notice that we defined as many d_1-open balls as d_2-open balls, which is as many x_i are defined by $\mathcal{I}$, which is finite. So for some $\varepsilon > 0$ there exists a number of d_1-open balls of radius ε that cover A. Since ε is arbitrary, this holds for any $\varepsilon > 0$. So A is d_1-totally bounded.

5. if there exists $c' > 0$ such that $c'd_2 \leq d_1 \leq cd_2$, then $A \subseteq X$ is d_1-totally bounded iff it is d_2-totally bounded.

   Let A be d_2-totally bounded. Since $\exists\, c > 0 : d_1 \leq c \cdot d_2$, it follows that A is d_1-totally bounded. For this reason, if A is not d_1-totally bounded, it cannot be d_2-totally bounded.

   Let A be d_1-totally bounded. Since $\exists\, c' > 0 : d_2 \leq \frac{1}{c} \cdot d_2$, it follows that A is d_2-totally bounded. For this reason, if A is not d_2-totally bounded, it cannot be d_1-totally bounded.

   So if $\exists\, c > 0, c' > 0 : c'd_2 \leq d_1 \leq cd_2$, A is d_1-totally bounded iff it is d_2-totally bounded.

Useful Theorems and Results

Diagonal of a Euclidean N-cube

Let C be a "cube" in a Euclidean space with side length $\alpha > 0$. That is, if $x \in \mathbb{R}^N$ is the "center" of C, then

$$C = \left[x_1 - \frac{\alpha}{2}, x_1 + \frac{\alpha}{2}\right] \times \left[x_2 - \frac{\alpha}{2}, x_2 + \frac{\alpha}{2}\right] \times ... \times \left[x_N - \frac{\alpha}{2}, x_N + \frac{\alpha}{2}\right]$$

Then the maximum distance from this center x is equal to

$$\begin{aligned}
\max_{y \in C} d(x,y) &= \max_{y \in C} \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \\
&= \max_{\left\{y_i \in \left[x_i - \frac{\alpha}{2}, x_i + \frac{\alpha}{2}\right]\right\}_{i=1}^{N}} \sqrt{\sum_{i=1}^{N} |x_i - y_i|^2} \\
&= \sqrt{\sum_{i=1}^{N} \left|x_i - x_i \pm \frac{\alpha}{2}\right|^2} \\
&= \sqrt{\sum_{i=1}^{N} \left|\pm \frac{\alpha}{2}\right|^2} \\
&= \frac{\alpha}{2} \sqrt{\sum_{i=1}^{N} 1} \\
&= \frac{\alpha}{2} \sqrt{N}
\end{aligned}$$

and corresponds to all the "corners" of this N-cube.

Riesz's Lemma

For (X, d) normed vector space (i.e. the metric d is a p-norm), $(S, d|_{S \times S})$ non-dense linear subspace of (X, d), and $0 < \varepsilon < 1$, there exists $x \in X$ *of unit norm* (i.e. $d(\mathbf{0}, x) = ||x||_p = 1$) such that $d(x, s) \geq 1 - \varepsilon, \forall s \in S$.

Pigeonhole Principle

For $n, m, k \in \mathbb{N}$ with $n = km + 1$, if we distribute n elements across m sets then at least one set will contain at least $k + 1$ elements.