Athens University of Economics and Business
*Department of Economics*

Postgraduate Program - MSc in Economic Theory
*Course: Mathematical Economics (Mathematics II)*
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# Problem Set 1

Metric functions and metric spaces

## Exercise 1

Is $d(x,y) = |x-y|$ a metric?

## Exercise 2

Is the function $d : X \times X \to \mathbb{R}$ such that $d(x,y) = |x-y|$, $\forall x, y \in X$ a metric on the non-empty set $X \subseteq \mathbb{R}$?

## Exercise 3

Suppose that $(Y, d)$ is a metric space. Let $f : X \to Y$ be an injection from $X$ to $Y$. Define $d_f : X \times X \to \mathbb{R}$ such that $d_f(x,y) = d(f(x), f(y))$, $\forall x, y \in X$. Is $(X, d_f)$ a metric space?

## Exercise 4

Study whether or not the following pairs of sets and functions constitute metric spaces:

1. $X \neq \varnothing$ and $d(x,y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}$, $\forall x, y \in X$, with $c > 0$ (Discrete distance)
2. $X = \mathbb{R}$ and $d(x,y) = |e^x - e^y|$, $\forall x, y \in X$ [Sutherland Ex. 5.4 (b)]
3. $X = \varnothing$ and $d(x,y) = |x-y|$, $\forall x, y \in X$
4. $X = \mathbb{R}$ and $d(x,y) = ln(|e^x - e^y|)$, $\forall x, y \in X$
5. $X = [-1, 1]$ and $d(x,y) = |x^2 - y^2|$, $\forall x, y \in X$
6. $X = \mathbb{R}$ and $d(x,y) = |x - y^3|$, $\forall x, y \in X$
7. $X = [0, 1]$ and $d(x,y) = |x-y|^2$, $\forall x, y \in X$

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8. $X = \mathbb{R}^N$ and $d(x,y) = \left(\sum_{i=1}^{N} |x_i - y_i|^p\right)^{\frac{1}{p}}$, $\forall x, y \in X$, with $p, N \in \mathbb{N}^*$ (Minkowski distance)

**Exercise 5**

For any metric space $(X, d)$ and $\forall x, y, z, w \in X$, show that:

1. $|d(x,z) - d(z,y)| \leq d(x,y)$ [O'Searcoid Theorem 1.1.2, Sutherland Ex. 5.1]

2. $|d(x,y) - d(z,w)| \leq d(x,z) + d(y,w)$ [O'Searcoid Q 1.2, Sutherland Ex. 5.2]

**Exercise 6**

Let $X$ be some non-empty set. Let $d_1$, $d_2$, and $d_s$ be distance functions on $X$ such that $d_s = d_1 + d_2$ Determine whether the following statements always hold (or under which conditions they could hold):

1. If $d_1$ and $d_2$ are metrics on $X$, $d_s$ is a metric on X.

2. If $d_1$ is a metric and $d_2$ a pseudo-metric on $X$, $d_s$ is a metric on X.

3. If $d_1$ and $d_2$ are pseudo-metrics on $X$, $d_s$ is a metric on X.

**Exercise 7**

Consider a finite index set $\mathcal{I} = \{1, 2, ..., n\}$ with $n \in \mathbb{N}^*$ and for each of its elements, $i$, the functional metric spaces $(\mathcal{B}(X_i, \mathbb{R}), d^i_{sup})$ with

$$d^i_{sup}(f_i, g_i) = \sup_{x \in X_i} |f_i(x) - g_i(x)|, \forall f_i, g_i \in \mathcal{B}(X_i, \mathbb{R})$$

Consider the product set $B_\Pi := \prod_{i \in I} \mathcal{B}(X_i, \mathbb{R})$ with $f := (f_i)_{i \in I} \in B_\Pi$ and the function $d_\Pi : B_\Pi \times B_\Pi \to \mathbb{R}$ such that

$$d_\Pi(f, g) = \max_{i \in \mathcal{I}} \sup_{x \in X_i} |f_i(x) - g_i(x)|, \forall f, g \in B_\Pi$$

Is $(B_\Pi, d_\Pi)$ a metric space?

**Exercise 8** [O'Searcoid Q 1.8]

Let $P(S)$ be the power set of a non empty set, $S$. Let the function $d : P(S) \times P(S) \to \mathbb{R}$ such that

$$d(A, B) = |(A \setminus B) \cup (B \setminus A)|, \forall A, B \in P(S)$$

be a function that gives the cardinality of the symmetric difference between two elements of $P(S)$ (i.e. subsets of $S$). Is $d$ a metric on $P(S)$?

**Exercise 9** [Sutherland Ex. 5.14]

Let $n$ be a positive natural number. The distance functions:

1. $d_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, $\forall x, y \in \mathbb{R}^n$ (Manhattan distance)
2. $d_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_2(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$, $\forall x, y \in \mathbb{R}^n$ (Euclidean distance)
3. $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that $d_\infty(x, y) = \max_{i=1}^{n} |x_i - y_i|$, $\forall x, y \in \mathbb{R}^n$ (Chebyshev distance)

are all metrics on $\mathbb{R}^n$. Show that the following functional inequalities hold:

$$d_\infty \le d_2 \le d_1 \le n \cdot d_\infty \le n \cdot d_2 \le n \cdot d_1$$

**Exercise 10**

Let $X$ be an $n \times m$ real matrix, with $n, m \in \mathbb{N}^*$ and $n > m$, such that $rank(X) = m$. Then $P_X = X(X'X)^{-1}X'$ is the projection matrix of $X$. Let $Y \subseteq \mathbb{R}^n$ be non-empty and $\hat{Y}$ be its projected image through $P_X$. Define $d_X : Y \times Y \to \mathbb{R}$ such that $d_X(x, y) = ||P_X \cdot x - P_X \cdot y||$, $\forall x, y \in Y$ (i.e. $d_X$ is the Euclidean norm of an $n$-dimensional real vector). Show that $(Y, d_X)$ is a pseudo-metric space.

(Hint: Consider the example of exercise 3. Under which conditions for $f$ is $(X, d_f)$ a pseudo-metric space?)

**Exercise 11**

Let $(X, d)$ be a metric space and consider a real function $f : \mathbb{R} \to \mathbb{R}$. Define $d' : X \times X \to \mathbb{R}$ such that $d'(x, y) = f(d(x, y))$, $\forall x, y \in X$.

1. Deduce the necessary conditions for $f$ so that $d'$ be a metric on $X$.
2. Let $f(x) = \dfrac{x}{1+x}$, $\forall x \in \mathbb{R}_+$. Is $d'$ a metric on $X$?
3. Let $f(x) = ln(1 + x)$, $\forall x \in \mathbb{R}_+$. Is $d'$ a metric on $X$?
4. Let $f(x) = x^\alpha$, $\forall x \in \mathbb{R}_+$ with $0 < \alpha < 1$. Is $d'$ a metric on $X$?
5. Let $f$ be a strictly increasing concave real function such that $f(0) = 0$. Is $d'$ a metric on $X$?

**Useful Theorems and Results**

**Cardinality and Set Operations**

Cardinality is a measure of the number of elements in a set. The following properties hold with respect to cardinality:

$$|\varnothing| = 0 \tag{1}$$

$$|A| + |B| = |A \cup B| + |A \cap B| \tag{2}$$

$$|A \setminus B| = |A| - |A \cap B| \tag{3}$$

**Square of the sum of $N$ numbers**

$$\left(\sum_{i=1}^{N} a_i\right)^2 = \sum_{i=1}^{N} a_i^2 + 2\sum_{i=1}^{N}\sum_{j=1}^{i-1} a_i a_j \tag{4}$$

**Hölder's inequality**

For all $x, y \in \mathbb{R}^N$ and $\alpha, \beta \in (1, +\infty)$ such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$, it holds that

$$\sum_{i=1}^{N} |x_i y_i| \leq \left(\sum_{i=1}^{N} |x_i|^\alpha\right)^{\frac{1}{\alpha}} \left(\sum_{i=1}^{N} |y_i|^\beta\right)^{\frac{1}{\beta}} \tag{5}$$

For $\alpha = \beta = 2$ we get the Cauchy-Schwartz inequality.

For $x \in \mathbb{R}^N$ we call $||x||_p \coloneqq \left(\sum_{i=1}^{N} |x_i|^p\right)^{\frac{1}{p}}$ the $p$-norm of $x$.