# Lecture 6

## Total Boundedness

# Total boundedness

Remember: by "reducing" any finite open (closed) cover to a single ball we have proven an equivalent definition of boundedness:

▷ $A \subseteq X$ is $d$-bounded iff $\exists\, \varepsilon > 0$ : $\exists$ a finite collection of open (closed) balls of radius $\varepsilon$ that covers $A$.

▷ What do we obtain if we transform the first existential quantifier ($\exists$) to a universal ($\forall$)

▷ We have to be careful: allow the finite collection characteristics [centers, cardinality] to depend on $\varepsilon$ in order to obtain something "general"

Definition. $A \subseteq X$ is ($d$-) totally bounded iff $\forall\, \varepsilon > 0$, $\exists$ a finite collection of open balls of radius $\varepsilon$, that covers $A$. ✓

May depend on $\varepsilon$

[Dually we can consider collections of closed balls].

Remark. The definition allows the dependence of $Z$ on $\varepsilon$. Hence both the centers and the cardinality of $Z$ is allowed to change with $\varepsilon$.

This is actually a helpful lemma

Remark. When $A$ is totally bounded then $\forall \varepsilon>0$ $Z_\varepsilon$ can be chosen as a subset of $A$ (i.e. the ball centers can be chosen to lie inside $A$).

May depend on $\varepsilon$

Proof. Suppose that $A$ is totally bounded. Let $\varepsilon>0$.

Due to the previous for $\delta=\varepsilon/2$ there exists a finite $Z$ such that the collection $\mathcal{O}(Z,\varepsilon/2)$ covers $A$. The collection $\mathcal{O}(Z,\varepsilon/2)$ is the set of open balls $\{O_d(x_1,\varepsilon/2), O_d(x_2,\varepsilon/2), \dots, O_d(x_n,\varepsilon/2)\}$ ($Z=\{x_1,x_2,\dots,x_n\}$ for $n$ that may depend on $\varepsilon/2$).

If $Z\subseteq A$ then the collection $\mathcal{O}(Z,\varepsilon)$ covers $A$ since $O_d(x_i,\varepsilon/2)\subseteq O_d(x_i,\varepsilon)$ $\forall i=1,\dots,n$. If $Z\nsubseteq A$ then:

Suppose that $x_1\notin A$ without loss of generality.

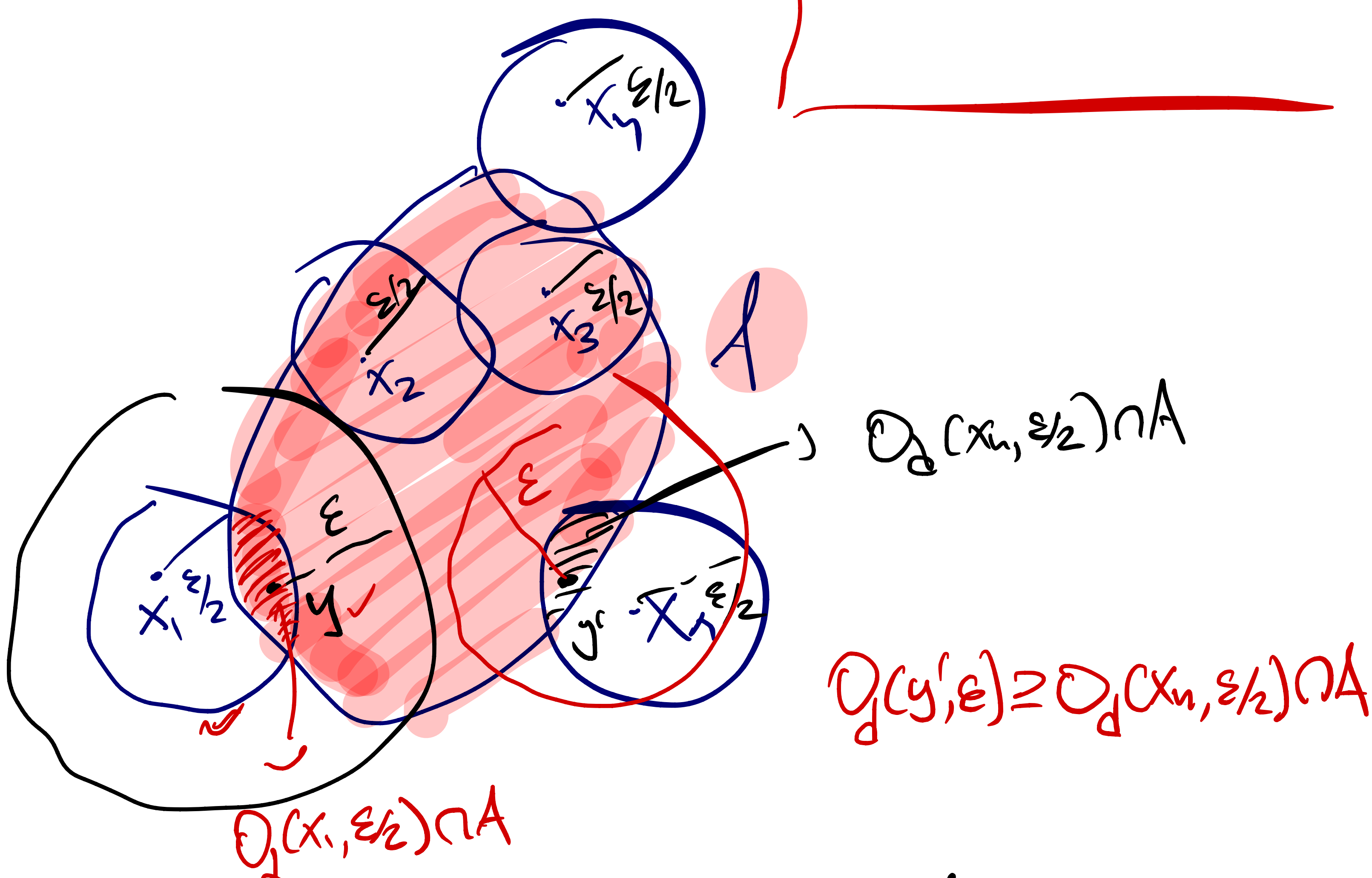
Since $\bigcup_{i=1}^{n} O_d(x_i,\frac{\varepsilon}{2}) \supseteq A$ we consider $O_d(x_1,\varepsilon/2)\cap A$. If this is empty then the ball $O_d(x_1,\varepsilon/2)$ is not needed in the covering and thereby $x_1$ can be discarded from $Z$ and we can move on to the next element of $Z$ that does not lie in $A$.

So suppose that $O_d(x_1,\varepsilon/2)\cap A$ is not empty. Let $y\in O_d(x_1,\varepsilon/2)\cap A$

Let $y\in O_d(x_1,\varepsilon/2)$. Consider $O_d(y,\varepsilon)$. We have that if $z\in O_d(x_1,\varepsilon/2)\cap A$ then $z\in O_d(y,\varepsilon)$.

[redacted]

the converse is obvious!!!




$$O_d(y,\varepsilon) \supseteq O_d(x_1,\varepsilon/2)\cap A$$

[redacted]

(Continue the proof of that when $A$ is t.b., the ball centers of the collections that cover it can be chosen to lie inside $A$, $\forall \varepsilon>0$).

Reminder: For $\varepsilon>0$, we considered $\mathcal{O}_{\varepsilon/2}(Z)$ a finite cover of open balls of radius $\varepsilon/2$ that covers $A$.

$Z=\{x_1, x_2, \dots, x_n\}$. We supposed that $x_1 \notin A$.

We considered the case $O_d(x_1, \varepsilon/2)\cap A$.

We have chosen $y \in O_d(x_1, \varepsilon/2)\cap A$, and considered the ball $O_d(y,\varepsilon)$: we have that $O_d(y,\varepsilon) \supseteq O_d(x_1,\varepsilon/2)\cap A$.

This is due to that: if $z \in O_d(x_1, \varepsilon/2)\cap A$, then

tr. ineq + symmetry

$$d(y,z) \le \underset{<\varepsilon/2}{d(x_1,y)} + \underset{<\varepsilon/2}{d(x_1,z)} < \varepsilon/2 + \varepsilon/2 = \varepsilon \Rightarrow$$

z arbitrary

$$d(y,z)<\varepsilon \iff z \in O_d(y,\varepsilon) \Rightarrow O_d(y,\varepsilon)\supseteq O_d(x_1,\varepsilon/2)\cap A$$

Obviously we can repeat the previous procedure for any $x_i \in Z$ such that $x_i \notin A$.

Hence for any $x_i \in Z$ and $x_i \notin A$, and $O_d(x_i,\varepsilon/2)\cap A \neq \emptyset$, we can find some $y_i \in A$ such that $O_d(y_i,\varepsilon) \supseteq O_d(x_i,\varepsilon/2)\cap A$.

For each $x_i \in Z$, for which $x_i \in A$ we consider $O_d(x_i,\varepsilon)$.

Obviously due to monotonicity we have that
$O_d(x_i,\varepsilon) \supseteq O_d(x_i,\varepsilon/2) \cap A$. Hence we have constructed a finite collection of open balls or radius $\varepsilon$ with:
i. centers inside $A$ and ii. the collection covers $A$.
The result follows since $\varepsilon$ is arbitrary. $\square$

**Lemma.** The previous hold if instead of covers by open balls we had considered covers of closed balls. ✓

**Proof.** (covers of open balls $\Rightarrow$ covers of closed balls)
Suppose $A$ is t.b ✓ (based on covers of open balls) and $\varepsilon>0$. There exists a cover of open balls, say,

$$\mathcal{O}_2(Z) = \{O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \ldots, O_d(x_n,\varepsilon)\}$$

such that $\bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq A$. We have that $O_d(x_i,\varepsilon) \subseteq O_d[x_i,\varepsilon]$ $\forall i=1,2,\ldots,n \Rightarrow \bigcup_{i=1}^{n} O_d[x_i,\varepsilon] \supseteq A$. Hence

$\{O_d[x_1,\varepsilon], O_d[x_2,\varepsilon], \ldots, O_d[x_n,\varepsilon]\}$ is the required collection. (Hence the definition of t.b. using open balls implies the definition of t.b. using closed balls)

(covers of closed balls $\Rightarrow$ covers of open balls)

Suppose that $A$ satisfies the definition of t.b. w.r.t. covers of closed balls. Let $\varepsilon>0$: we know that there exists a finite collection of closed balls

$$\{ O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \ldots, O_d(x_n,\varepsilon) \}$$
$$\{ O_d[x_1,\varepsilon/2], O_d[x_2,\varepsilon/2], \ldots, O_d[x_n,\varepsilon/2] \}$$ that covers $A$. We know that $O_d(x_i,\varepsilon) \supseteq O_d[x_i,\varepsilon/2]$ $\forall i=1,\ldots,n$

$\Rightarrow$ $\bigcup_{i=1}^{n} O_d(x_i,\varepsilon) \supseteq \bigcup_{i=1}^{n} O_d[x_i,\varepsilon/2] \supseteq A$ hence the collection $\{ O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \ldots, O_d(x_n,\varepsilon) \}$ is the required one. Since $\varepsilon$ is arbitrary the result follows. $\square$

Further useful results: (analogous with the case of boundedness)

- $X, d$ will be considered totally bounded iff $X$ is totally bounded w.r.t. as a subset of itself.
- $A \subseteq X$ is totally bounded w.r.t. $d$, iff $A, d_A$ is totally bounded (Proof - Exercise! Use the result on the choice of the ball centers - it is similar to the proof of the analogous result for boundness).

Lemma If $A$ is totally bounded w.r.t. $d$ then it is also bounded w.r.t. $d$.

Proof. [ Check the definitions using the covers! ]

Lemma. If $A$ is finite then it is totally bounded. (this holds for any metric!)
↳ universally

Proof. If $A=\emptyset$ then $A$ is trivially t.b. Suppose that $A\neq\emptyset$, i.e. it is of the form

$$A=\{x_1, x_2, \dots, x_m\}, \quad x_i\in X \ \forall i=1,\dots,m$$

For $\varepsilon>0$, consider $\{O_d(x_1,\varepsilon), O_d(x_2,\varepsilon), \dots, O_d(x_m,\varepsilon)\}$

Since $x_i\in O_d(x_i,\varepsilon)$ $\forall i=1,\dots,m$, we have that

$$\bigcup_{i=1}^{m} O_d(x_i,\varepsilon)\supseteq A.$$ Since $\varepsilon$ is arbitrary

the result follows. $\square$

Lemma If $A$ is t.b. w.r.t. $d$ and $B\subseteq A$ then $B$ is also t.b. w.r.t. $d$.

Proof. $\forall\,\varepsilon>0$ the cover of open balls covers $A$ it necessarily covers $B$.

Dually: if $A$ is not t.b. w.r.t. $d$ and $B\supseteq A$ then als $B$ is not t.b. w.r.t. $d$

($A$ is not t.b. iff $\exists \varepsilon > 0$: every finite collection of open balls of radius $\varepsilon$ cannot cover $A$).

# Examples and Counter-examples

1. $X, d_d$ ($X, d_d$ is always bounded)

We will show that: $X, d_d$ is totally bounded iff it is finite.

↳ contrast this with boundedness

Proof a. (Finite $\Rightarrow$ t.b) (see above).

b. (if $X, d_d$ is t.b. then it is finite).

Suppose that $X$ is infinite. Let $\varepsilon = 1/2$.

(Reminder $O_{d_d}(x, 1/2) = \{x\}$ $\forall x \in X$). This directly implies that the only way to cover $X$ by open balls of radius $1/2$ is: $\bigcup_{x \in X} O_{d_d}(x, 1/2)$

$$\bigcup_{x \in X} O_{d_d}(x,1/2) = \bigcup_{x \in X} \{x\} = X$$

but the collection $\{O_{d_d}(x, 1/2), x \in X\}$ is not finite since $X$ is infinite. The result follows. $\square$

Hence the previous implies via example that total boundness is strictly stronger than boundness.
↳ the "curious" discrete spaces can also be useful in uncovering the "strength" of "similar" notions

2. Is $\mathbb{R}, d_u$ totally bounded? No since it is not even bounded. The same holds for $\mathbb{R}^d, d_I$.

(the interesting question here is when $A \subseteq \mathbb{R}^d$ is t.b. w.r.t. $d_I$)

Such counterexamples are easy to construct: e.g. $(\mathbb{R}^n, d_I)$, $(\mathbb{R}^n, d_{max})$ $(\mathbb{R}^n, d_1)$ are not bounded ⇒ they are not totally bounded w.r.t. the respective metrics.

3. Remember $\underline{B(\mathbb{N}, \mathbb{R})}, d_{sup}$ (bounded real sequences with the uniform metric).

$$A = \left\{ f_m \in B(\mathbb{N}, \mathbb{R}),\ f_m(n) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases},\ m \in \mathbb{N} \right\}$$

↳ $e_m$ (apologies for the multi-notation!)

Reminder: infinite dimensional vector representation of $f_m$

$$(0, 0, \ldots, 1, 0, \ldots 0, \ldots) //$$

$(m+1)^{th}$ position

e.g. $f_0$ $\quad (1, 0, 0, \ldots, 0, \ldots)$ ✓

We have proven that $A$ is bounded w.r.t.

$d_{sup}$ (we have used uniform boundness).

Is $A$ totally bounded w.r.t. $d_{sup}$? $A$ is not totally bounded due to that: we know that if $A$ were t.b. then the centers of the covering balls can be chosen to belong to $A$.

Also we have that $d_{sup}(f_n, f_m) = \begin{cases} 0, & n=m \\ 1, & n\neq m. \end{cases}$

$A$ is obviously infinite. For $\varepsilon = 1/2$

$O_{d_{sup}}(f_m, 1/2) \cap A = \{f_m\}$. Hence in order to be able to cover $A$ with open balls of the form $O_{d_{sup}}(f_m, 1/2)$, $m \in \mathbb{N}$ we need as many of those balls as there are elements of. But $A$ is infinite. Hence $A$ is not totally bounded w.r.t. $d_{sup}$. $\rhd$ → Again: one more instance of the relative strength bound./t. bound.

In order to construct examples the following notion will be usefull:

End of lecture 6.

$$\rhd \; d_{sup}(e_i, e_j) = \sup_{n\in\mathbb{N}} |e_i(n) - e_j(n)| = \sup_{n\in\mathbb{N}} \left| \begin{cases} 1, & i=n \\ 0, & i\neq n \end{cases} - \begin{cases} 1, & j=n \\ 0, & j\neq n \end{cases} \right| = \begin{cases} 0, & i\neq j \\ 1, & i\neq j \end{cases}$$

$e_i = (0, \ldots, \underset{i^{th}\text{ pos.}}{1}, 0, \ldots)$, $e_j = (0, \ldots, \underset{i\text{th pos}}{0}, \ldots, \underset{j\text{th pos}}{1}, 0, \ldots 0, \ldots)$

# Covering Numbers and Metric Entropy

[Our approach: Not very rigorous]

We will use without loss of generality covers by closed balls.

Framework: $X, d$, $\underset{\neq \emptyset}{A} \subseteq X$, $\varepsilon > 0$

**Definition:** The covering number of $A$ w.r.t. $\varepsilon$ and $d$, $N(\varepsilon, d, A) :=$ Min # of closed balls of radius $\varepsilon$ that cover $A$.

Remark: it is possible to prove that $N(\varepsilon, d, A)$ is unique $\forall \varepsilon, d, A$.
↳ well defined as a function of $\varepsilon, d, A$

$\ln N(\varepsilon, d, A)$ is the Metric entropy number w.r.t. $\varepsilon$ and $d$.

- It is easy to show that $A$ is t.b. w.r.t. $d$ iff $N(\varepsilon, d, A) \in \mathbb{N}$ (i.e. it is $< +\infty$) $\forall \varepsilon > 0$.

(dually $A$ is not t.b. w.r.t. $d$ iff $\exists \varepsilon > 0$: $N(\varepsilon, d, A) = +\infty$)

- It is easy to see that $N(\varepsilon, d, A)$ is decreasing w.r.t. $\varepsilon$ for fixed $d, A$.

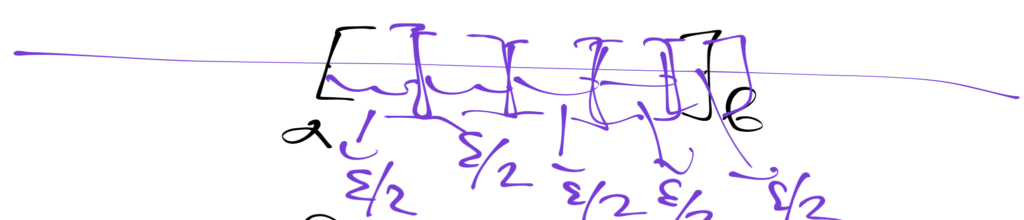
- <u>Generally</u>, as $\varepsilon \downarrow 0$ $\left(\begin{smallmatrix}\varepsilon \to 0 \\ \varepsilon > 0\end{smallmatrix}\right)$, $N(\varepsilon, d, A) \to +\infty$ for fixed $d, A$

even when $A$ is t.b.

- What is very informative on the complexity of $A$ as a metric (sub-)space is the rate at which $N(\varepsilon, d, A)$ (or equivalently $\ln N(\varepsilon, d, A)$) diverges as $\varepsilon \downarrow 0$.

E.g. $X = \mathbb{R}$, $d = d_u$ $\qquad [a, b] = O_{d_u}\left[\frac{a+b}{2}, \frac{b-a}{2}\right]$

closed ball




$$N(\varepsilon, d_u, [a, b]) = \begin{cases} 1, & \varepsilon \geq \frac{b-a}{2} \\ \text{ο μικρότερος φυσικός} \geq \frac{b-a}{2\varepsilon}, & \varepsilon < \frac{b-a}{2} \end{cases} < +\infty$$

$\forall \varepsilon > 0$. Hence $[a, b]$ is t.b. w.r.t. $d_u$

Hence we have shown that $\forall x \in \mathbb{R}, \varepsilon > 0$, $O_{d_u}[x, \varepsilon]$ is totally bounded w.r.t. $d_u$. Hence if $A \subseteq \mathbb{R}$ is bounded w.r.t. $d_u$ then it is also t.b. w.r.t. $d_u$.

Hence Boundness $\Longleftrightarrow$ Total Boundness inside $\mathbb{R}, d_u$

[redacted]

- If $(X,d)$ a metric space, $A \subseteq X$, $\varepsilon > 0$

$N(\varepsilon, d, A) :=$ the "smallest" number of closed balls of $\varepsilon$ radius that covers $A$. [Covering number of $A$ corre-sponding to $\varepsilon$].

- Variations of the above are definable if we use open balls and/or if we restrict the centers to lie in $A$. Those variations will not necessarily equal each other but it is possible to prove that they have equivalent asymptotic behavior as $\varepsilon \downarrow 0$, hence they convey the same information on the complexity of $A$.
- $A$ is t.b. iff $N(\varepsilon, d, A) \in \mathbb{R}$ $\forall \varepsilon > 0$.
- $\ln N(\varepsilon, d, A)$ is termed as metric entropy number of $A$.

E.g. $X = \mathbb{R}$, $d = d_u$, $A = [\alpha, \beta] = O_{d_u}[\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}]$
$\alpha, \beta \in \mathbb{R}$

we have easily shown $N(\varepsilon, d_u, [\alpha, \beta]) \sim \frac{c}{\varepsilon}$ depends on $\alpha, \beta$

Since $\frac{c}{\varepsilon} \in \mathbb{R}$ $\forall \varepsilon > 0$ this essentially showed us that boundness is equivalent to total boundness in $\mathbb{R}, d_u$

- Metric Entropy: $\ln N(\varepsilon, d_u, [\alpha, \beta]) \sim \ln c - \ln \varepsilon$ dominant term as $\varepsilon \downarrow 0$

$\sim -\ln \varepsilon$

E.g. $X = \mathbb{R}^d$ $d = d_I$ it is possible to prove that:
$\forall x \in \mathbb{R}^d, \delta > 0$, $A = O_{d_I}[x, \delta]$, $N(\varepsilon, d_I, O_{d_I}[x, \delta]) \sim \frac{c}{\varepsilon^d}$

for some suitable $c > 0$. Since $\frac{c}{\varepsilon^d} \in \mathbb{R}$ $\forall \varepsilon > 0$ every closed ball in this metric space is totally bounded.

Hence boundness is equivalent to total boundness in $\mathbb{R}^d, d_I$.

— Metric Entropy: $\ln N(\varepsilon, d_I, O_{d_I}[x,\delta]) \sim -d\ln\varepsilon$

E.g. $X = B([0,1],\mathbb{R})$, $d = d_{sup}$, $A = \{f:[0,1]\to\mathbb{R},\ f(0)=0,$
$L>0$ $\quad |f(x)-f(y)| \le L|x-y|\}$

We know that $A$ is bounded (uniform boundness)

but is it totally bounded?

We can show that $N(\varepsilon, d_{sup}, A) \sim \exp(c/\varepsilon)$ and

$c$ is a positive constant independent of $\varepsilon$ that depends on $L$.

Since $\exp(c/\varepsilon)$ is finite $\forall \varepsilon>0$ we conclude that $A$ is totally bounded w.r.t. $d_{sup}$. By comparing the asymptotic behavior of $e^{c/\varepsilon}$ with $c/\varepsilon^d$ we can conclude that the particular $A$ is "more complex" than any Euclidean ball

## Metric Comparison and total boundness

$X$, $d_1, d_2$ well defined metrics, for some $c>0$, $d_1 \le c d_2$.

$\Leftrightarrow \forall x\in X, \varepsilon>0 \quad O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x, c\varepsilon) \Leftrightarrow \forall x\in X, \forall \delta>0$

$\delta = c\varepsilon$ bijection since $c>0$.

(*) $O_{d_2}(x, \delta/c) \subseteq O_{d_1}(x,\delta)$

Suppose that $A$ is t.b. w.r.t. $d_2$. In order to see whether $A$ is t.b. w.r.t. $d_1$, let $\delta>0$, since $A$ is $d_2$ t.b. we have that there exists a finite cover of open balls or radius $\delta/c$ of $A$ w.r.t. $d_2$, hence due to (*) there exists a finite cover of open balls of radius $\delta$ of $A$ w.r.t. $d_1$ (Just use the centers of the previous cover). Hence since $\delta$ is arbitrary $A$ is t.b. w.r.t. $d_1$. Hence we have proven:

**Lemma.** Total boundness w.r.t. $d_2 \Rightarrow$ Total boundness w.r.t. $d_1$.

Furthermore due to (*) we have that

$$N(\delta/c, d_2, A) \geq N(\delta, d_1, A).$$

**Lemma.** Suppose that for $c_1, c_2 > 0$: $c_1 d_2 \leq d_1 \leq c_2 d_2$. (**)

Then Total boundness w.r.t. $d_1 \Leftrightarrow$ Total boundness w.r.t. $d_2$.

For example in $\mathbb{R}^d$ boundness is equivalent to total boundness w.r.t. any of the metrics $d_I, d_{max}, d_{II}$. (*)

(Exercise: provide the details)

Exercise: what is the relation between the respective covering numbers when (**) holds?

- Total boundness can be very useful in applications related with the issue of convergence in function spaces.

[redacted]

it seems that t.b. "matters" in function spaces

(*) Implies that in $(\mathbb{R}^d, d_*)$ tot. boundedness $\Leftrightarrow$ boundedness $\forall * = I, II, max$

Finally: What about total boundedness in Metric subspaces?

if $A$ (as a subset of $X$) is totally bounded then $\forall \varepsilon > 0$ $\exists$ $\{O_d(x_i, \varepsilon), x_i \in A, i = 1, \dots, n(\varepsilon)\}$ that covers $A$

$$\Downarrow \quad \Uparrow$$

$\exists$ $\{O_d(x_i, \varepsilon) \cap A, x_i \in A, i = 1, \dots, n(\varepsilon)\}$ covers $A$ $\forall \varepsilon > 0$

$\hookrightarrow O_{d_A}(x_i, \varepsilon)$

$$\Downarrow \quad \Uparrow$$

$\forall \varepsilon > 0, \exists \{O_{d_A}(x_i, \varepsilon), i = 1, \dots, n(\varepsilon)\}$ covers $A \Rightarrow$

$$\Downarrow \quad \Uparrow$$

$$\boxed{A, d_A \text{ is totally bounded}}$$

Hence:

**Lemma.** $A$ $(\neq \emptyset)$ is ($d$-) totally bounded iff the Metric space $A, d_A$ is totally bounded

(We can simply restrict our attention to whole Metric spaces).

* When is $A \subseteq X$ not totally bounded (w.r.t. d)?

$\Leftrightarrow$ Deny the property: "$\forall \varepsilon > 0$, $\exists$ a finite collection of open balls of radius $\varepsilon$ that covers A"

$\Leftrightarrow$ $\exists \varepsilon > 0$, ($\forall$) every finite collection of open balls of radius $\varepsilon$ cannot cover A

necessary condition for total boundednes

* total boundedness $\Rightarrow$ boundedness $\Leftrightarrow$

A is not bounded $\Rightarrow$ A is not totally bounded

* From the denial of the definition of total boundedness we obtain:

if $A \subseteq B$ and A is not totally bounded

$\Rightarrow$ B is not totally bounded

Hence dually to the previous lemma, "failure" of total boundedness is inherited by the supersets.