Lecture 3

- Further Examples - Function Spaces
- Metric Subspaces
- Open and Closed Balls

We move on with an important (in what follows) "category" of examples:

Example: X is a set of functions

For $Y \neq \emptyset$ consider $B(Y,\mathbb{R})$ and let

$d_{sup}: B(Y,\mathbb{R}) \times B(Y,\mathbb{R}) \to \mathbb{R}$ be defined

by:

$$f,g \in B(Y,\mathbb{R}), \quad d_{sup}(f,g) := \sup_{x\in Y} |f(x)-g(x)|$$

* d_{sup} is termed uniform metric

Notice that:

tr. ineq. for abs. value + monoton. of sup

$$* \quad \overset{d_{sup}(f,g)}{\overset{\shortparallel}{\sup_{x\in Y} |f(x)-g(x)|}} \leq \sup_{x\in Y}\left(|f(x)|+|g(x)|\right)$$

$$\leq \underbrace{\sup_{x\in Y} |f(x)|}_{+\infty \text{ (no: } < +\infty)} + \sup_{x\in Y} |g(x)| < +\infty$$

$f,g \in B(Y,\mathbb{R})$

Hence d_{sup} is a well defined real function.

* Obviously $\sup_{x\in Y} |f(x)-g(x)| \geq 0 \ \forall f,g$

hence p.d. holds ✓

* $0 = d_{sup}(f,g) \Leftrightarrow 0 = \sup_{x\in Y} |f(x)-g(x)|$

$\Leftrightarrow |f(x)-g(x)| = 0 \ \forall x \in Y \Leftrightarrow f(x) = g(x) \ \forall x \in Y$

$\Leftrightarrow f = g$ hence separation holds

* $d_{sup}(g,f) = \sup_{x \in Y} |g(x) - f(x)| = \sup |f(x) - g(x)|$
symmetry of $|\cdot - \cdot|$

$= d_{sup}(f,g)$, hence symmetry holds.

* if $f, g, h \in B(Y, \mathbb{R})$

$d_{sup}(f,g) = \sup_{x \in Y} |f(x) - g(x)| = \sup_{x \in Y} |f(x) \pm h(x) - g(x)|$ ✓

$= \sup_{x \in Y} |(f(x) - h(x)) + (h(x) - g(x))| \leq \sup_{x \in Y} [|f(x) - h(x)| +$

$+ |h(x) - g(x)|] \leq \sup_{x \in Y} |f(x) - h(x)| + \sup_{x \in Y} |h(x) - g(x)|$

$= d_{sup}(f,h) + d_{sup}(h,g)$ hence the triangle inequality holds. ∎

* The above is quite general and important in what follows. Some further sub-examples

arbitrary element of $\mathbb{R}^n$ $\to$ $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

* (Sub)-Example: It is possible to perceive $(\mathbb{R}^n, d_{max})$ as a special case of $(B(Y,\mathbb{R}), d_{sup})$ for a suitable choice of Y:

- Let $Y = \{1, 2, \dots, n\}$. A function $f: Y \to \mathbb{R}$ can be represented by the n-vector $(\underset{\in\mathbb{R}}{f(1)}, \underset{\in\mathbb{R}}{f(2)}, \dots, \underset{\in\mathbb{R}}{f(n)})$. Inversely if $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ then it defines a function $f: Y \to \mathbb{R}$ by $f(i) = x_i$, $i = 1, \dots, n$. Hence there exists a bijective correspondence between $\{f: Y \to \mathbb{R}\}$ and $\mathbb{R}^n$.
- Furthermore any $f: Y \to \mathbb{R}$ is bounded since Y is finite (remember the first lecture - its image in $\mathbb{R}$ is finite). Hence $B(Y, \mathbb{R}) \cong \mathbb{R}^n$.
- If $x, y \in \mathbb{R}^n$, $d_{max}(x,y) = \max\limits_{i=1,\dots,n} |x_i - y_i|$

$$= \max_{i=1,\dots,n} |f(i) - g(i)| = \max_{i \in Y} |f(i) - g(i)|$$

$f: Y \to \mathbb{R}$ represents x

$g: Y \to \mathbb{R}$ represents y

Max exists $\Rightarrow$ equals sup

$$= \sup_{i \in Y} |f(i) - g(i)|$$

$$= d_{sup}(f, g).$$

Hence d_{max} "is essentially" d_{sup} and thus

$$(\mathbb{R}^n, d_{max}) \approx (B(\{1,2,\dots,n\}, \mathbb{R}), d_{sup}) \text{ - III}$$

* (Sub-Example) - Bounded real sequences

- A real sequence is an "infinite dimensional vector" of the form $(x_0, x_1, \dots, x_n, \dots)$, $x_n \in \mathbb{R}, n \in \mathbb{N}$

initial element

as many elements as the elements of $\mathbb{N}$

no final element

e.g. α. $(0, 1, 2, \dots, n, \dots)$ γ. $(1, 1, 1, \dots, 1, \dots)$

β. $(1, 1/2, \dots, 1/(n+1), \dots)$

- Equivalently a real sequence $(x_0, x_1, \dots, x_n, \dots)$ is represented by a function $f: \mathbb{N} \to \mathbb{R}$ since:

  - $(x_0, x_1, \dots, x_n, \dots)$ defines $f: \mathbb{N} \to \mathbb{R}$ with

    $f(n) := x_n$

  - $f: \mathbb{N} \to \mathbb{R}$ defines the vector

    $(f(0), f(1), \dots, f(n), \dots)$

  a. $f(n) = n$, b. $f(n) = \frac{1}{n+1}$

- A real sequence is bounded iff the $f: \mathbb{N} \to \mathbb{R}$ that represents it is a bounded real function

$\overset{\text{1st lecture}}{\iff} \sup_{n \in \mathbb{N}} |f(n)| < +\infty \iff \boxed{\sup_{n \in \mathbb{N}} |x_n| < +\infty}$

  a. $\sup_{n \in \mathbb{N}} |n| = +\infty$ not bounded

  b. $\sup_{n \in \mathbb{N}} |1/_{n+1}| = 1 < +\infty$ bounded.

- Hence $(B(\mathbb{N}, \mathbb{R}), d_{sup})$ is the space of bounded real sequences equipped with the uniform metric.

e.g. c. $x_n = 1 \;\; \forall n \in \mathbb{N}$ $\quad (1, 1, \ldots, 1, \ldots)$ → constant sequence at 1

$f(n) = 1$ → $\in B(\mathbb{N}, \mathbb{R})$

$$\sup_{n\in\mathbb{N}} |x_n| = \sup_{n\in\mathbb{N}} 1 = 1 < +\infty$$

$$d_{\sup}\left(\left(1, \tfrac{1}{2}, \ldots, \tfrac{1}{n+1}, \ldots\right), (1, 1, \ldots, 1, \ldots) \right)$$

$$= \sup_{n\in\mathbb{N}} \left| \frac{1}{n+1} - 1 \right| = \sup_{n\in\mathbb{N}} \left| \frac{n}{n+1} \right| = \sup_{n\in\mathbb{N}} \frac{n}{n+1}$$

$$\underset{\text{why?}}{=} \lim_{n\to+\infty} \frac{n}{n+1} = 1. \quad \square$$

* (Sub)-Example

$X = [0,1]$, and consider $B([0,1], \mathbb{R})$.

Notice that $B([0,1], \mathbb{R}) \neq \emptyset$ since it contains the constant functions $f_c(x) = C$, $\forall x \in [0,1]$, $c \in \mathbb{R}$.

Consider also $L > 0$, and the set

$$A_L := \{ f : [0,1] \to \mathbb{R} : f(0) = 0, \; \forall x, y \in [0,1] \; |f(x) - f(y)| \leq L |x - y| \}.$$

$$\left|\frac{\partial g}{\partial x}(z)\right| = \frac{L}{e}e^z$$

Notice that $A_L \neq \emptyset$ since for example

$$\underset{\in[0,1]}{g(x)} := \frac{L}{e}(e^x - 1) \in A_L :$$

$$g(x) = g(y) + \frac{\partial g}{\partial x}(y^*)(x-y)$$

y^* between x, y

$$g(x) - g(y) = \frac{\partial g}{\partial x}(y^*)(x-y)$$

$$\Downarrow$$

$$|g(x)-g(y)| = \left|\frac{\partial g}{\partial x}(y^*)\right| |y-x| \le \sup_{y^*\in[0,1]} \left|\frac{\partial g}{\partial x}(y^*)\right| |y-x|$$

$$g(0) = \frac{L}{e}(e^0 - 1) = 0$$

if $x, y \in [0,1]$, $|g(x) - g(y)| = \frac{L}{e}|e^x - e^y|$

$$\le \frac{L}{e} \sup_{z\in[0,1]} \underset{= e^1}{e^z} |x-y| = L|x-y|$$

MVT

Furthermore if $f \in A_L$ we have that

$$\sup_{x\in[0,1]} |f(x)| = \sup_{x\in[0,1]} |f(x) - f(0)| \le \sup_{x\in[0,1]} L|x-0|$$

$$= L \sup_{x\in[0,1]} |x| = L < +\infty \text{ hence } f \in B([0,1], \mathbb{R})$$

Thereby $\emptyset \neq A_L \subseteq B([0,1], \mathbb{R})$

A_L is an example of a uniformly Lipschitz set of functions

[to be defined...]

Hence $B([0,1], \mathbb{R})$ does not only contains constant functions.

Another family of functions inside $B([0,1],\mathbb{R})$ is $A_{\mathbb{N}} := \{ g_n : [0,1] \to \mathbb{R},\ g_n(x) := nx,\ n \in \mathbb{N}\}$ since

$$\sup_{x\in[0,1]} |g_n(x)| = \sup_{x\in[0,1]} |nx| = \sup_{x\in[0,1]} nx \underset{n\geq 0}{=} n \sup_{x\in[0,1]} x = n < \infty$$

hence $g_n \in B([0,1],\mathbb{R}) \Rightarrow A_{\mathbb{N}} \subseteq B([0,1],\mathbb{R})$.

Exe. $d_{sup}\left(\frac{1}{e}(e^x - 1), nx\right)$, $\forall n \in \mathbb{N}$.

$(B([0,1],\mathbb{R}), d_{sup})$ is the resulting metric space when equipped with the uniform metric. $\boxtimes$

B. Metric Subspaces

Definition. Suppose that (X,d) is a metric space and $\emptyset \neq Y \subseteq X$. Let d_Y denote the restriction of d on $Y \times Y$ (i.e. consider d restricted only to pairs of elements of Y). Obviously

d_Y is a well-defined metric (why?) that can equip Y, and the metric space (Y, d_Y) is termed a metric subspace of (X, d).

Examples:

- $(\mathbb{R}^n, d_{11})$, $Y = \{x \in \mathbb{R}^n : x_i = 0 \text{ or } 1, \forall i = 1, \dots, n\}$, $\mathbb{R}^n = X$, $Y \subseteq \mathbb{R}^n$, $Y \neq \emptyset$

(obviously $Y \subseteq \mathbb{R}^n$). If $x, y \in Y$, then

$$d_{11}(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} \begin{cases} 1, & x_i \neq y_i \\ 0, & x_i = y_i \end{cases}$$

$$= \#\{i = 1, 2, \dots, n : x_i \neq y_i\} = d_H(x,y)$$

Hence when d_{11} is restricted on Y it reduces to d_H. Thus (Y, d_H) is a metric subspace of $(\mathbb{R}^n, d_{11})$ [Does this hold for d_{I_2}, d_{max}?]

- (A_L, d_{sup}), (A_{IN}, d_{sup}) are metric subspaces of $(B([0,1], \mathbb{R}), d_{sup})$. $A_L \neq \emptyset$, $A_{IN} \neq \emptyset$

abuse of notation

- $([0,1], d_u)$, $([0,1], d_e)$ are metric subspaces of $(\mathbb{R}, d_u)$ and $(\mathbb{R}, d_e)$ respectively.

- etc. (we will see more examples as the course progresses)

[The notion of the metric subspace is an example of the construction of a metric space from a given one. We will later on see further examples, like product spaces, etc.]

C. Open and Closed Balls.

The notions of open and closed balls in a metric space is fundamental. It generalizes the notion of open and closed (bounded) intervals in $\mathbb{R}$, thus systems of such balls represent a lot of information about the metric space:

Definition: Let (X,d) a metric space, $x \in X$ and $\varepsilon > 0$. The open ball of (X,d) centered at x, with radius ε is

$$O_d(x,\varepsilon) := \{ y \in X : d(x,y) < \varepsilon \}.$$

Respecetively the closed ball of (X,d) centered at x with radius ε is

$$O_d[x,\varepsilon] := \{ y \in X : d(x,y) \le \varepsilon \}. \quad \boxed{\equiv}$$

* $\forall x \in X, \varepsilon > 0$, $x \in O_d(x,\varepsilon)$ and $x \in O_d[x,\varepsilon]$ since due to (2) $d(x,x) = 0 \le \varepsilon$.

* if $y \in X$ and $d(x,y) < \varepsilon \Rightarrow d(x,y) \le \varepsilon$ hence $\Rightarrow y \in O_d[x,\varepsilon]$

  $y \in O_d(x,\varepsilon) \Rightarrow$ $\forall x \in X, \varepsilon > 0$ $\quad O_d(x,\varepsilon) \subseteq O_d[x,\varepsilon]$

  Not Necessarily strict

* if $\delta > \varepsilon$ then if $d(x,y) < \varepsilon$ (or $\le \varepsilon$) $\Rightarrow d(x,y) < \delta$ hence $\Rightarrow y \in O_d(x,\delta)$

  $y \in O_d(x,\varepsilon)$, $y \in O_d[x,\varepsilon]$

  $$O_d(x,\varepsilon) \subseteq O_d(x,\delta)$$

Putting the above together, $\forall x \in X$, $\varepsilon, \delta > 0$, $\varepsilon < \delta$

$$* \quad O_d(x,\varepsilon) \subseteq O_d[x,\varepsilon] \subseteq O_d(x,\delta) \subseteq O_d[x,\delta]$$

Not Necessarilly strict inclusions

$$d_1(x,y) = \begin{cases} 0, & x=y \\ 1, & x\neq y \end{cases}$$

Examples:

- (X, d_1) discrete space

We have that (why?)

$$O_d(x,\varepsilon) = \begin{cases} \{x\}, & \varepsilon \leq 1 \\ X, & \varepsilon > 1 \end{cases}$$

$$O_d[x,\varepsilon] = \begin{cases} \{x\}, & \varepsilon < 1 \\ X, & \varepsilon \geq 1 \end{cases}$$

→ A peculiar example - balls are either singletons at the centers or the whole space!

- $(\mathbb{R}, d_u)$

We have that

$$O_d(x,\varepsilon) = (x-\varepsilon, x+\varepsilon)$$

$$O_d[x,\varepsilon] = [x-\varepsilon, x+\varepsilon]$$

the "usual" intervals.

– $(\mathbb{R}, d_e)$ [remember $d_e(x,y) = |e^x - e^y|$]

Notice that $d_e(x,y) \le \varepsilon \iff |e^x - e^y| \le \varepsilon \iff$

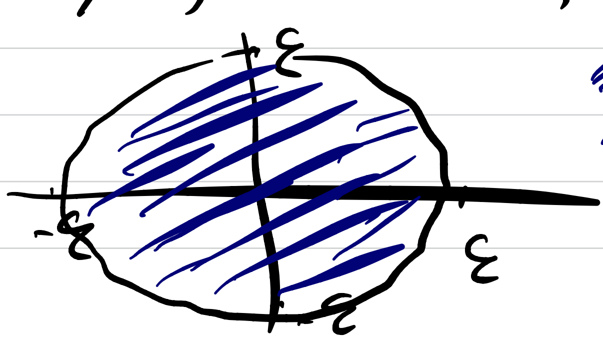
$-\varepsilon \le e^x - e^y \le \varepsilon \iff e^x - \varepsilon \le e^y \le e^x + \varepsilon$ ✓

$\iff y \in (k, \ln(e^x+\varepsilon))$ with $k = \begin{cases} \ln(e^x - \varepsilon), & e^x > \varepsilon \\ -\infty, & e^x \le \varepsilon \end{cases}$

$y \in \begin{matrix} [\ln(e^x-\varepsilon), \ln(e^x+\varepsilon)], & k > -\infty \\ (-\infty, \ln(e^x+\varepsilon)], & k = -\infty \end{matrix}$

Hence in $(\mathbb{R}, d_e)$ intervals of the form $(-\infty, c)$ or $(-\infty, c]$ are considered of finite radius something that is obviously not the case in $(\mathbb{R}, d_u)$.

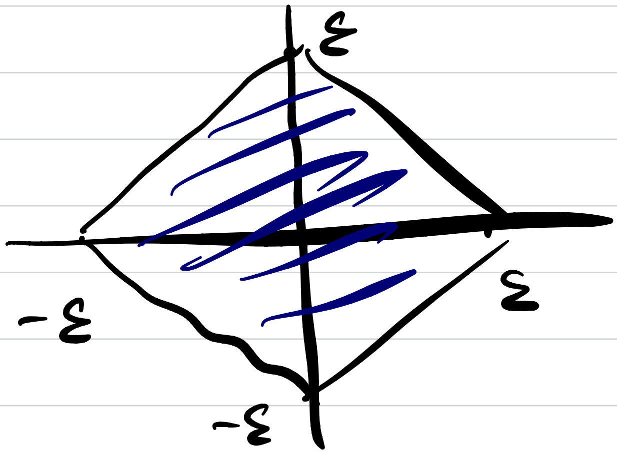
– $(\mathbb{R}^2, d_I)$, $x = 0_{2\times1}$, $\varepsilon > 0$



$= O_{d_I}(0_{2\times1}, \varepsilon)$

$\cup\, O = O_{d_I}[0_{2\times1}, \varepsilon]$

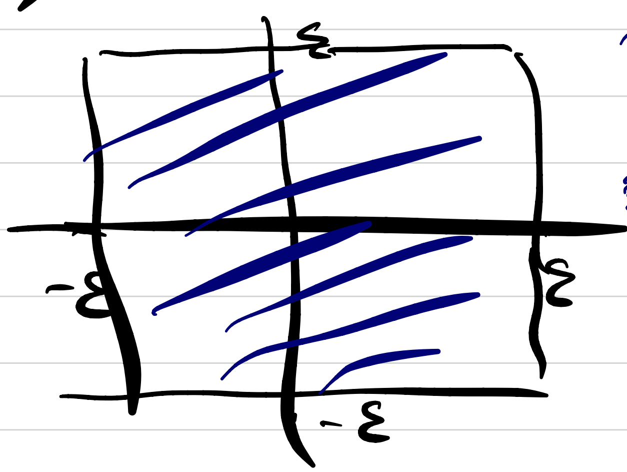
$-\ (\mathbb{R}^2, d_{11})$ $\quad x = 0_{2\times 1},\quad \varepsilon > 0$



$\equiv = O_{d_{11}}(0_{2\times 1}, \varepsilon)$

$\equiv \cup \lozenge = O_{d_{11}}[0_{2\times 1}, \varepsilon]$

$-\ (\mathbb{R}^2, d_{max})$ $\quad x = 0_{2\times 1},\quad \varepsilon > 0$



$\equiv = O_{d_{max}}(0_{2\times 1}, \varepsilon)$

$\equiv \cup \square = O_{d_{max}}[0_{2\times 1}, \varepsilon]$

Hence different metrics can attribute different geometric objects as balls on the same carrier set.

Exercise: Consider $(B([0,1],\mathbb{R}), d_{sup})$. Let $f(x) = 1$ $\forall x \in [0,1]$. "Draw" $O_{d_{sup}}(f,1)$ and $O_{d_{sup}}[f,1]$. $\blacksquare$

C1. An initial Property of Separation via Balls.

Lemma [Separation]. Let $x, y \in X$. Then $x \neq y$ $\iff \exists\, \varepsilon_1, \varepsilon_2 > 0 : O_d(x,\varepsilon_1) \cap O_d(y,\varepsilon_2) = \emptyset$.

Proof. ($\Leftarrow$) obvious from that $x \in O_d(x,\varepsilon_1)$, $y \in O_d(y,\varepsilon_2)$ and disjointness.

($\Rightarrow$) $x \neq y \overset{(2)}{\iff} d(x,y) := \varepsilon > 0$. Let $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$.

Consider $O_d(x,\varepsilon/2)$, $O_d(y,\varepsilon/2)$ and suppose that $z \in O_d(x,\varepsilon/2) \cap O_d(y,\varepsilon/2)$. Hence $d(x,z) < \varepsilon/2$ and $d(y,z) < \varepsilon/2$. Then

$$\varepsilon = d(x,y) \underset{\substack{\text{tr. ineq.}\\ +\text{symmetry}}}{\leq} \underbrace{\overset{<\varepsilon/2}{d(x,z)} + \overset{<\varepsilon/2}{d(y,z)}}_{} < \underbrace{\varepsilon/2 + \varepsilon/2}_{=\varepsilon}$$

impossible $\blacksquare$

Exercise: Does something analogous hold for closed balls? (obviously yes)

Hence balls separate points

[We could generalize in the obvious way the construction of balls for pseudo-metrics. Would such a separation result hold generally then?]

C_{II}. Balls and dominance

Consider (X,d_1), (X,d_2), $x\in X$, $\varepsilon>0$ and $O_{d_1}(x,\varepsilon)$, $O_{d_2}(x,\varepsilon)$. Suppose that $\exists c>0$: $d_1 \leq c\, d_2$ [functional inequality]

Let $y \in O_{d_2}(x,\varepsilon) \iff d_2(x,y)<\varepsilon \overset{c>0}{\iff} c\,d_2(x,y)<c\varepsilon$

$\overset{\text{dom.}}{\Rightarrow} d_1(x,y) \in O_{d_1}(x,c\varepsilon) \Rightarrow O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x,c\varepsilon)$.

Hence $\forall x\in X, \varepsilon>0$, $d_1 \leq c\,d_2 \Rightarrow O_{d_2}(x,\varepsilon) \subseteq O_{d_1}(x,c\varepsilon)$

antitone transformation. $\boxtimes$