Lecture 2

- Further Examples
- Dominance Relations

Further Examples

* Remember that:

- A Metric space is a pair (X, d) where

$X \neq \emptyset$, $d : X \times X \rightarrow \mathbb{R}$ such that:

Metric

Carrier

i. $\forall x, y \in X$, $d(x,y) \geq 0$ (p.d.)

ii. $d(x,y) = 0 \Leftrightarrow x = y$ (sep.)

iii. $\forall x, y \in X$, $d(x,y) = d(y,x)$ (sym.)

iv. $\forall x, y, z \in X$, $d(x,y) \leq d(x,z) + d(z,y)$ (tr. ineq.)

- When only $(\Leftarrow)$ in ii then $\Rightarrow$ pseudo-metric

pseudo metric space

- Every $X \neq \emptyset$ can be equipped with the discrete metric $\Rightarrow$ the collection of metric spaces is "vast"

Exe: $c \in \mathbb{R}$, define $d_c(x,y) = \begin{cases} 0, & x = y \\ c, & x \neq y \end{cases}$

- For what c is d_c a (pseudo)-metric?
- Is d_c related to d_1?

- It is possible that the same carrier can be equipped with more than one metric. □

Example: Hamming Distance - Information Theory

$n \in \mathbb{N}^*$, $X = \{0,1\}^n = \{ (x_1, x_2, \ldots, x_n), x_i \in \{0,1\}$ $\forall i = 1, \ldots, n \}$ $\subseteq \mathbb{R}^n$

↳ 0-1 n-dimensional vectors

bits of information

E.g. $n = 2$, $X = \{(0,0), (0,1), (1,0), (1,1)\}$ etc

$d_H : X \times X \to \mathbb{R}$ is defined by:

$x, y \in X$, $d_H(x,y) := \#\{ i = 1, \ldots, n : x_i \neq y_i \}$

"number of disagreements between x, y"

$n = 2$

$d_H\left(\binom{0}{0}, \binom{1}{1}\right) = 2$

$d_H\left(\binom{0}{0}, \binom{0}{1}\right) = 1$

- $0 \leq d_H(x,y) \leq n$ $\forall x, y \in X$

hence d_H is a well defined real function and p.d. holds

- $0 = d_H(x,y) \iff$ x and y nowhere disagree $\iff x = y$ [hence separation holds]

- $d_H(x,y) = \#\{i=1,\dots,n : x_i \neq y_i\} = \#\{i=1,\dots,n, y_i \neq x_i\}$
$= d_H(y,x)$ [hence symmetry holds]

- if $x,y,z \in X$ then

$\#[\{i=1,\dots,n : x_i \neq z_i\} \cup \{i=1,\dots,n : z_i \neq y_i\}]$ consider the union

$\leq \#\{i=1,\dots,n : x_i \neq z_i\} + \#\{i=1,\dots,n : z_i \neq y_i\} =$

$= d_H(x,z) + d_H(z,y)$ (a)

and

since it can be that $x_i \neq z_i$ and $z_i \neq y_i$ but $x_i = y_i$

$\{i=1,\dots,n : x_i \neq y_i\} \subseteq \{i=1,\dots,n : x_i \neq z_i\} \cup$
$\{i=1,\dots,n : z_i \neq y_i\} \Rightarrow \#\{i=1,\dots,n : x_i \neq y_i\} = d_H(x,y)$

$\leq \#[\{i=1,\dots,n : x_i \neq z_i\} \cup \{i=1,\dots,n : z_i \neq y_i\}]$ (b)

(a), (b) $\Rightarrow$ $\boxed{d_H(x,y) \leq d_H(x,z) + d_H(z,y)}$

[hence tr. ineq. holds since x, y, z are arbitrary]

$\rightarrow d_H$ Useful in error correction in message transmissions, etc. $\square$

Example: Metrics via P.d. Matrices

$n \in \mathbb{N}^*$, $X = \mathbb{R}^n$, A is an $n \times n$ strictly positive definite (symmetric) matrix $\Leftrightarrow$

$\forall x \in \mathbb{R}^n$ $x'Ax \geq 0$ with equality iff $x = 0_{n\times 1}$

↳ column or row vector

$\Leftrightarrow$ all the eigenvalues of A are strictly positive

real due to symmetry

e.g. $A = \underbrace{c\, id_{n\times n}}$, $c > 0$, $id_{n\times n} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & 0 & \ddots & 1 \end{pmatrix}$

$\underline{d_A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}}$ is defined by

$$d_A(x,y) := \left((x-y)' A \underbrace{(x-y)}_{\in \mathbb{R}^n}\right)^{1/2}$$

– Since n is finite and A is positive definite $0 \leq d_A(x,y) < +\infty$ $\forall x,y \in \mathbb{R}^n$

hence d_A is a well-defined real valued function and p.d. holds.

- $\underbrace{0 = d_A(x,y)}$ $\Leftrightarrow$ $0 = \underbrace{\left((x-y)'A(x-y)\right)^{1/2}}$

$\Leftrightarrow$ $\underline{0 = (x-y)'A(x-y)}$ $\overset{A \text{ is strictly p.d.}}{\Leftrightarrow}$ $\underline{x-y = 0_{n\times 1}}$

$\Leftrightarrow \forall z \in \mathbb{R}^n \quad z'Az \geq 0$ with equality iff $z = 0_{n\times 1}$

$\Leftrightarrow$ $x = y$ hence separation holds.

- $d_A(y,x) = \left((y-x)'A(y-x)\right)^{1/2} = \left([-(x-y)]'A[-(x,y)]\right)^{1/2} = \left((-1)^2(x-y)'A(x-y)\right)^{1/2} =$

$= \left((x-y)'A(x-y)\right)^{1/2} = d_A(x,y)$

hence symmetry holds

- Exogenous: Cauchy-Schwarz inequality

if A is p.d. then $|x^{*\prime}Ay^*| \leq (x^{*\prime}Ax^*)^{1/2}(y^{*\prime}Ay^*)^{1/2}$, $\forall x^*, y^* \in \mathbb{R}^n$ ✓

Not necessarilly strictly positive definite.

Now if $x, y, z \in \mathbb{R}^n$ then

$$d_A^2(x,y) = (x-y)' A (x-y) = (x \pm z - y)' A (x \pm z - y)$$

$$= \left((x-z)+(z-y)\right)' A \left((x-z)+(z-y)\right) =$$

linearity of transp.

$$= \left((x-z)'+(z-y)'\right) A \left((x-z)+(z-y)\right) =$$

$$= (x-z)'A(x-z) + \underbrace{(x-z)'A(z-y)}_{(\alpha)} + \underbrace{(z-y)'A(x-z)}_{(\beta)} + (z-y)'A(z-y)$$

$$\left[(z-y)'A(x-z)\right]' = \left((x-z)'A(z-y)\right)' = (x-z)'A(z-y)$$

$(\alpha)' = (\beta)$ but $(\alpha), (\beta)$ are $1 \times 1 \Rightarrow (\alpha) = (\beta)$

$$= (x-z)'A(x-z) + (z-y)'A(z-y) + 2(x-z)'A(z-y)$$

$$= d_A^2(x,z) + d_A^2(z,y) + 2(x-z)'A(z-y)$$

$$\leq d_A^2(x,z) + d_A^2(z,y) + 2\left|(x-z)'A(z-y)\right|$$ C.S ineq.

$$\underset{\text{C.S. ineq.}}{\leq} d_A^2(x,y) + d_A^2(z,y) + 2\left((x-z)'A(x-z)\right)^{1/2}\left[(z-y)'A(z-y)\right]^{1/2}$$

$x^* = x - z$ $\quad$ $d_A(x,z)$

$y^* = z - y$ $\quad$ $d_A(z,y)$

$$= d_A^2(x,z) + d_A^2(z,y) + 2\, d_A(x,z)\, d_A(z,y)$$
$$= (d_A(x,z) + d_A(z,y))^2$$

Hence we have shown:

$$d_A^2(x,y) \leq \underbrace{\left(d_A(x,z) + d_A(z,y)\right)^2}$$ $\Rightarrow$ Monotonicity of $\sqrt{\cdot}$

$$d_A(x,y) \leq d_A(x,z) + d_A(z,y)$$ [Hence the triangle inequality holds]

* Show that when A is "only" positive definite then d_A is a pseudometric.

* When $A = id_{n \times n}$ $\quad d_I(x,y) = \left((x-y)' I (x-y)\right)^{1/2}$

$$= \left((x-y)'(x-y)\right)^{1/2} = \left((x_1-y_1, \dots, x_n-y_n)\begin{pmatrix} x_1-y_1 \\ \vdots \\ x_n-y_n \end{pmatrix}\right)^{1/2}$$

$$= \Big(\underbrace{\sum_{i=1}^{n}(x_i-y_i)^2}_{\text{sum of squared deviations}}\Big)^{1/2} \sim \text{Euclidean distance}$$

and $(\mathbb{R}^n, d_I)$ is the "usual" <u>Euclidean Metric</u> space.

* When $A = c\, id_{n\times n}$, $c>0$ then

$$d_A(x,y) = \left((x-y)' c I (x-y)\right)^{1/2} =$$

$$= \sqrt{c}\left((x-y)'(x-y)\right)^{1/2} = \sqrt{c}\, d_I(x,y)$$

[hence it is possible that different Metrics on the same X can obey relations between them]

* When $n=1$, $\mathbb{R}^n = \mathbb{R}$, $A = c$ with $c>0$

and $\underline{d_A(x,y)} = \left((x-y)' c (x-y)\right)^{1/2} =$

$$= \sqrt{c(x-y)^2} = \sqrt{c}\,|x-y|$$

when $c=1$ (and thereby $A = id_{1\times 1}$)

$$d_I(x,y) = |x-y| = d_u(x,y)$$

✓

And thereby the Euclidean Metric is an "extension" of d_u on "higher dimensions".

$\square$

Example: Metric defined by the sum of absolute deviations

$X = \mathbb{R}^n$ (as before), consider

$d_{11} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$x,y \in \mathbb{R}^n$, $d_{11}(x,y) = \sum_{i=1}^{n} |x_i - y_i| = \sum_{i=1}^{n} d_u(x_i, y_i)$

- Obviously $0 \leq d_{11}(x,y) < +\infty \quad \forall x,y \in \mathbb{R}^n$

hence d_{11} is a well defined real function and p.d. holds.

- $0 = d_{11}(x,y) \Leftrightarrow 0 = \sum_{i=1}^{n} \underbrace{|x_i - y_i|}_{\geq 0 \ \forall i=1,\dots,n} \Leftrightarrow$

$|x_i - y_i| = 0 \ \forall i=1,\dots,n \Leftrightarrow x_i = y_i \ \forall i=1,\dots,n \Leftrightarrow x=y$.

hence separation holds.

$\to d_u(x_i, y_i) = 0 \ \forall i=1,\dots,n$

$$- d_{11}(y,x) = \sum_{i=1}^{n} |y_i - x_i| \overset{= |x_i - y_i| \ \forall i=1,\dots,n}{=} \sum_{i=1}^{n} |x_i - y_i| =$$

$$= d_{11}(x,y) \text{ hence symmetry holds}$$

$$- \text{ if } x,y,z \in \mathbb{R}^n, \ d_{11}(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

$$= \sum_{i=1}^{n} |x_i \pm z_i - y_i| = \sum_{i=1}^{n} |(x_i - z_i) + (z_i - y_i)|$$

once again we "interpolate" z

use triangle ineq. for absolute value

$$\leq \sum_{i=1}^{n} (|x_i - z_i| + |z_i - y_i|) = \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i|$$

tr. ineq. for absolute value

$$= d_{11}(x,z) + d_{11}(z,y)$$

hence the triangle inequality holds

* When $n=1$ $d_{11}(x,y) = |x-y| = d_u(x,y)$
hence d_{11} is also an extension of d_u to "higher dimensions".

* eg. $n=2$ $\quad d_{11}\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right) =$

$$= |2-1| + |1-2| = 2$$

$$d_I\left(\begin{pmatrix}2\\1\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}\right) = \sqrt{(2-1)^2+(1-2)^2} =$$

$$= \sqrt{2}$$

hence $d_I \neq d_{11}$.

$\square$

Example: Metric defined via the Maximum absolute deviation

$X = \mathbb{R}^n$ (as before)

$d_{max} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by

$x,y \in \mathbb{R}^n$, $d_{max}(x,y) := \max\limits_{i=1,\dots,n} |x_i - y_i| = \max\limits_{i=1,\dots,n} d_u(x_i, y_i)$

– Since n finite, $0 \leq \max\limits_{i=1,\dots,n} |x_i - y_i| < +\infty$

hence d_{max} is a well defined real function that also satisfies p.d.

- $0 = d_{max}(x,y) \iff 0 = \max_{i=1,\dots,n} |x_i - y_i| \iff$ $\geq 0 \ \forall i=1,\dots,n$

$\to d_u(x_i,y_i) \iff x_i = y_i \ \forall i=1,\dots,n$

$|x_i - y_i| = 0 \ \forall i = 1,\dots,n \iff x = y$ hence separation holds.

$\to$ abs value symmetric

- $d_{max}(y,x) = \max_{i=1,\dots,n} |y_i - x_i| = \max_{i=1,\dots,n} |x_i - y_i|$

$= d_{max}(x,y)$ hence symmetry holds

- if $x,y,z \in \mathbb{R}^n$, $d_{max}(x,y) = \max_{i=1,\dots,n} |x_i - y_i|$

$= \max_{i=1,\dots,n} |x_i \pm z_i - y_i| \leq \max_{i=1,\dots,n} \left(|x_i - z_i| + |z_i - y_i|\right)$

$\max_{i=1,\dots,n} |(x_i - z_i) + (z_i - y_i)|$

remember the properties of sup + tr. ineq. for the abs. value

$\leq \max_{i=1,\dots,n} |x_i - z_i| + \max_{i=1,\dots,n} |z_i - y_i| = d_{max}(x,z) + d_{max}(z,y)$

again property of sup

hence the triangle inequality holds.

* When $n=1$, $d_{max}(x,y) = \max |x-y| = |x-y|$

$= d_u(x,y)$ hence d_{max} is yet another extension of d_u

$* \; n=2, \quad d_{max}\left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right) =$

$$= \max\{|2-1|, |1-2|\} = \max\{|1|, |1|\}$$

$$= \underline{\underline{1}} \neq d_{11}\left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right) = 2$$

$$\neq d_I\left(\begin{pmatrix}2\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix}\right) = \sqrt{2}$$

More generally for $n > 1$

$$(\mathbb{R}^n, d_I) \neq (\mathbb{R}^n, d_{11})$$

$$\neq (\mathbb{R}^n, d_{max}) \neq$$

However Notice: $\forall x, y \in \mathbb{R}^n$

(a) $- \; d_{max}(x,y) = \max|x_i - y_i| \leq \sum_{i=1}^{n} |x_i - y_i| = d_{11}(x,y)$

(b) $- \; d_{11}(x,y) = \sum_{i=1}^{n} |x_i - y_i| \leq \sum_{i=1}^{n} \max_{i=1,\dots,n} |x_i - y_i|$

$$= n \cdot \max_{i=1,\dots,n} |x_i - y_i| = n \, d_{max}(x,y)$$

(c) $- \; d^2_{max}(x,y) = (\max|x_i - y_i|)^2 = \max\left(|x_i - y_i|^2\right) = \max(x_i - y_i)^2$

$$\leq \sum_{i=1}^{n} (x_i - y_i)^2 = d_I^2(x,y) \Rightarrow d_{max}(x,y) \leq d_I(x,y)$$

Monotonicity of $\sqrt{\cdot}$

(δ) $-$ $d_I^2(x,y) = \sum_{i=1}^{n}(x_i-y_i)^2 \le \sum_{i=1}^{n} \max(x_i-y_i)^2$

$\le n \cdot \max(x_i-y_i)^2 = n\left(\max|x_i-y_i|\right)^2$

$= n\, d_{max}^2(x,y) \xRightarrow[\text{of } \sqrt{\cdot}]{\text{mon.}} d_I(x,y) \le \sqrt{n}\, d_{max}(x,y)$

Hence by (α), (β)

$d_{max} \le d_{11}$ ✓

$d_{11} \le n\, d_{max}$ ✓

(functional ineq. hold $\forall x,y$)

$d_{max} \le d_{11} \le n\, d_{max}$

by (γ), (δ)

$d_{max} \le d_I$

$d_I \le \sqrt{n}\, d_{max}$

$d_{max} \le d_I \le \sqrt{n}\, d_{max}$

by (β), (γ)

$d_{11} \le n\, d_I$

by (α), (δ)

$d_I \le \sqrt{n}\, d_{11}$

$\frac{1}{n} d_{11} \le d_I \le \sqrt{n}\, d_{11}$

Hence the three metrics obey functional relations!

* More generally for d_1, d_2 metrics defined on X:

i. d_2 dominates d_1 iff ✓

$\exists\, c > 0$: $d_1 \leq c\, d_2$ [functional inequality $\Leftrightarrow d_1(x,y) \leq c\, d_2(x,y)\ \forall x,y \in X$]

ii. d_1 is equivalent to d_2 iff ✓

d_2 dominates d_1 and d_1 dominates d_2

Exercise: d_1 is equivalent to d_2 iff $\exists\, c^*, C^* > 0$: $c^* d_2 \leq d_1 \leq C^* d_2$

Hence d_I, d_{II}, and d_{max} are pairwise equivalent

* i.e. different metrics on the same X can obey dominance relations [based on functional inequalities]

→ We will later on the course see what this implies in terms of properties (E)

End of lecture 2.

Example : function Space

For $Y \neq \emptyset$ consider $B(Y, \mathbb{R})$ and let

$d_{sup} : B(Y,\mathbb{R}) \times B(Y,\mathbb{R}) \to \mathbb{R}$ be defined

by:

$$f,g \in B(Y,\mathbb{R}),\quad d_{sup}(f,g) := \sup_{x\in Y} |f(x)-g(x)|$$

Notice that:

$$* \quad \sup_{x\in Y} |f(x)-g(x)| \leq \sup_{x\in Y}(|f(x)|+|g(x)|)$$

$$\leq \underbrace{\sup_{x\in Y} |f(x)|}_{<+\infty} + \underbrace{\sup_{x\in Y} |g(x)|}_{<+\infty} < +\infty$$

$f,g \in B(Y,\mathbb{R})$

Hence dsup is a well defined real function.

$*$ Obviously $\sup_{x\in Y} |f(x)-g(x)| \geq 0 \ \forall f,g$

hence p.d. holds

$$* \quad 0 = d_{sup}(f,g) \Leftrightarrow 0 = \sup_{x\in Y} |f(x)-g(x)|$$

$$\Leftrightarrow |f(x)-g(x)| = 0 \ \forall x \in Y \Leftrightarrow f(x) = g(x) \ \forall x \in Y$$

$\Leftarrow)$ $f=g$ hence separation holds

* $d_{sup}(g,f) = \sup_{x\in Y} |g(x)-f(x)| = \sup |f(x)-g(x)|$
symmetry of $|\cdot - \cdot|$
$= d_{sup}(f,g)$, hence symmetry holds.

* if $f,g,h \in B(Y,\mathbb{R})$

$d_{sup}(f,g) = \sup_{x\in Y} |f(x)-g(x)| = \sup_{x\in Y} |f(x) \pm h(x) - g(x)|$

$= \sup_{x\in Y} |(f(x)-h(x)) - (h(x)-g(x))| \leq \sup_{x\in Y} \big[|f(x)-h(x)| +$

$+ |h(x)-g(x)| \big] \leq \sup_{x\in Y} |f(x)-h(x)| + \sup_{x\in Y} |h(x)-g(x)|$

$= d_{sup}(f,h) + d_{sup}(h,g)$ hence the triangle inequality holds.

TO BE CONTINUED...