Lecture 18/05/20

Reminder:

* (x_n) is $(d-)$ Cauchy iff, $\forall \varepsilon>0\ \exists n^*(\varepsilon)$:
  $\forall n,m \geq n^*(\varepsilon)\quad d(x_n,x_m)<\varepsilon$

$\Leftrightarrow \forall \varepsilon>0\ \exists n^*(\varepsilon)\ \forall n,m\geq n^*(\varepsilon)\ d(x_n,x_m)\in O_{d_{\mathbb{R}}}(0,\varepsilon)$

$\Leftrightarrow \lim\limits_{\min(n,m)\to\infty} d(x_n,x_m)=0$

* If $\exists\, c>0 : d_1 \leq c d_2$ and (x_n) is d_2-Cauchy then

since $d_1(x_n,x_m)\leq c\, d_2(x_n,x_m)\ \forall n,m \Rightarrow$

$\lim\limits_{\min(n,m)\to+\infty} d_1(x_n,x_m) \leq c \lim\limits_{\min(n,m)\to\infty} d_2(x_n,x_m)=0 \Rightarrow$

(x_n) is d_1- Cauchy.

Hence Set of d_1-Cauchy Seq $\supseteq$ Set of d_2-Cauchy Seq.

Continue:

- If $\exists\, c_1,c_2>0$: $c_1 d_2 \leq d_1 \leq c_2 d_2$ then (x_n) is d_1- Cauch iff (x_n) is d_2-Cauchy

[i.e. Set of d_1-Cauchy Seq = Set of d_2-Cauchy Seq]

$\Downarrow$ + result on the limits

X is d_1- complete $\Leftrightarrow$ X is d_2- complete $\square$

E.g. $X = \mathbb{R}^p$ $d = d_I$ it is possible to prove that $\mathbb{R}^p$ is d_I - complete. Hence $\mathbb{R}^p$ is d-complete for any $d = d_I, d_{max}, d_{11}$.

2. Lipschitz Continuity

* "Usual" continuity of functions between metric spaces is somewhat equivalent to the "preservation of sequential convergence".
* Is "Cauchyness" preserved by usual continuity?

Answer: No! For example: $X = (0,1]$, $d_X = d_u$ (restricted); $Y = \mathbb{R}$, $d_Y = d_u$. $f: (0,1] \to \mathbb{R}$, $f(x) := \frac{1}{x}$ (f is obviously d_u/d_u - continuous). For $x_n = \frac{1}{n+1}$, $n \in \mathbb{N}$, we know that $(1/_{n+1})$ is d_u-Cauchy in X. Now, $f(x_n) = \frac{1}{x_n} = \frac{1}{1/_{n+1}} = n+1$ hence we obtain $(f(x_n)) = (n+1)$ "living" in Y. The latter is not d_u-Cauchy. Hence the "usual" continuity does not generally preserve Cauchyness.

* Is it possible to define stronger notions of continuity that "preserve" Cauchyness?

One way to do it is through the notion of Lipschitz continuity.

Framework: (X, d_X), (Y, d_Y), $f: X \to Y$.

Definition. f is (d_Y/d_X-) Lipschitz continuous iff $\exists c > 0$:

$$\forall x, y \in X, \quad d_Y(f(x), f(y)) \leq c\, d_X(x, y).$$

Lemma. If f is (d_Y/d_X-) Lipschitz continuous then it is continuous.

Proof. Let $x, x_n \in X$, and $x_n \to x$ (w.r.t. d_X). It suffices to prove that $f(x_n) \to f(x)$ (w.r.t. d_Y), which is equivalent to that $\lim_{n \to +\infty} d_Y(f(x_n), f(x)) = 0$.

Since f is Lipschitz-continuous

$$d_Y(f(x_n), f(x)) \leq c\, d_X(x_n, x) \quad \forall n \in \mathbb{N} \implies$$

$$\lim_{n \to +\infty} \underbrace{d_Y(f(x_n), f(x))}_{\geq 0 \ \forall n \in \mathbb{N}} \leq c \lim_{n \to +\infty} d_X(x_n, x) = 0 \quad \text{since } x_n \to x$$

$$\implies \lim_{n \to +\infty} d_Y(f(x_n), f(x)) = 0. \quad \square$$

Remark. If c exists it is not unique, for any $c^* > c$ we have that $\forall x, y \in X$, $d_Y(f(x), f(y)) \leq c^* d(x, y)$

the infimum of positive constants for which the definitional inequality holds and it is called Lipschitz Coefficient of f (C_f). $\square$

Does Lipschitz Continuity preserve Cauchyness?

Lemma. If (x_n) is (d_X-) Cauchy and f is Lipschitz continuous, then $(f(x_n))$ is (d_Y-) Cauchy.

Proof. Since f is Lipschitz continuous,

$$d_Y(f(x_n), f(x_m)) \leq C_f\, d_X(x_n, x_m) \quad \forall n,m \in \mathbb{N} \Rightarrow$$

$$\lim_{\min(n,m)\to+\infty} \underset{\overset{\vee\!|}{0}}{\underline{d_Y(f(x_n), f(x_m))}} \leq C_f \lim_{\min(n,m)\to+\infty} \underline{d_X(x_n, x_m)} = 0 \Rightarrow$$

(since (x_n) is Cauchy)

$\lim_{\min(n,m)\to+\infty} d_Y(f(x_n), f(x_m)) = 0$, hence $(f(x_n))$ is (d_Y-) Cauchy. $\square$

Corollary. "Usual" continuity does not imply Lipschitz-continuity.

Proof. "Usual continuity" does not generally preserve Cauchyness but due to the previous Lemma, Lipschitz continuity preserves Cauchyness. $\square$

Hence due to the previous Lipshitz-Continuity is genuinely stronger than usual continuity

Examples:

A. (X,d) a m.s., with $(\mathbb{R}, d_u)$. Let $z \in X$. Define

$$f_z : X \to \mathbb{R}, \text{ as } f_z(x) := d(z,x)$$

Let $x,y \in X$, $d_u(f_z(x), f_z(y)) = |f_z(x) - f_z(y)| =$

$$= |d(z,x) - d(z,y)| \leq d(x,y)$$

dual version of triangle inequality

Hence we have shown that

$$\forall x,y \in X \quad d_u(f_z(x), f_z(y)) \leq d(x,y)$$

hence f_z is d_u/d - Lipschitz continuous with Lipschitz coefficient equal to 1, and this holds $\forall z \in X$.

– Generally Metrics are Lipschitz continuous!

B. $X = \mathbb{R}^q$, $d_X = d_I$, $Y = \mathbb{R}^p$, $d_Y = d_I$.

in $\mathbb{R}^q$ $\quad$ $\mathbb{R}^p$

Consider the following: α. Euclidean Norm in $\mathbb{R}^q$

$$x \in \mathbb{R}^q, \quad \|x\| := \left(\sum_{i=1}^{q} x_i^2\right)^{1/2}$$

$$x = (x_1, x_2, \dots, x_q)$$

For $x,y \in \mathbb{R}^q$: $\quad d_I(x,y) = \|x-y\|$ ✓

b. The Euclidean norm in $\mathbb{R}^p$ is analogous.

c. Let A be a $p \times q$ real matrix, i.e.

$$A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1q} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2q} \\ \vdots & & \ddots & \\ \alpha_{p1} & \alpha_{p2} & \dots & \alpha_{pq} \end{pmatrix}$$

then the Frobenius norm of A is $\|A\|_F = \left(\sum_{i=1}^{p} \sum_{j=1}^{q} \alpha_{ij}^2 \right)^{1/2}$.

d. $x \in \mathbb{R}^q \Rightarrow \underset{p \times q\ q\times 1}{Ax} \in \mathbb{R}^p$

It is possible to prove that $\|\cdot\|$ and $\| \|_F$ satisfy the following sub-multiplicative property:

$$\|Ax\| \leq \|A\|_F \|x\|$$

Euclidean norm in $\mathbb{R}^p$

Euclidean norm in $\mathbb{R}^q$.

e. Let $f: \mathbb{R}^q \to \mathbb{R}^p$ that is everywhere differentiable in $\mathbb{R}^q$, which implies that for any $x \in \mathbb{R}^q$, the Jacobian $\dfrac{\partial f(x)}{\partial x'}$ (i.e. the matrix defined as $f(x) \in \mathbb{R}^p$ $\forall x \in$, $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_p(x) \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \end{pmatrix}$

$$\frac{\partial f}{\partial x'}(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x), & \frac{\partial f_1}{\partial x_2}(x), & \dots, & \frac{\partial f_1}{\partial x_q}(x) \\ \frac{\partial f_2}{\partial x_1}(x), & \frac{\partial f_2}{\partial x_2}(x), & \dots, & \frac{\partial f_2}{\partial x_q}(x) \\ \vdots & & & \\ \frac{\partial f_p}{\partial x_1}(x), & \frac{\partial f_p}{\partial x_2}(x), & \dots, & \frac{\partial f_p}{\partial x_q}(x) \end{pmatrix}_{p \times q}$$

is well defined $\forall x \in \mathbb{R}^q$. This is equivalent to that $0 \leq \| \frac{\partial f(x)}{\partial x'} \|_F < +\infty$ $\forall x \in \mathbb{R}^q$ (i.e. $x \to \| \frac{\partial f(x)}{\partial x'} \|_F$ is a well defined function $\mathbb{R}^q \to \mathbb{R}$). Iff this $x \to \| \frac{\partial f(x)}{\partial x'} \|_F$ mapping is bounded then f is termed everywhere differentiable with bounded derivative. (Remember that this is equivalent to that $\sup_{x \in \mathbb{R}^q} \| \frac{\partial f}{\partial x'}(x) \|_F < +\infty$).

Theorem. If $f : \mathbb{R}^q \to \mathbb{R}^p$ that is everywhere differentiable with bounded derivative, then f is $(d_I / d_I -)$ Lipschitz continuous, with $C_f = \sup_{x \in \mathbb{R}^q} \| \frac{\partial f}{\partial x'}(x) \|_F$.

Euclidean Metric in $\mathbb{R}^p$

Euclidean Metric in $\mathbb{R}^q$

Proof. Let arbitrary $x, y \in \mathbb{R}^q$. We have that

$$d_I(f(x), f(y)) = \| f(x) - f(y) \|$$

Euclidean Metric in $\mathbb{R}^p$

Since f is everywhere differentiable we have due the Mean value theorem that

$(*)$ $f(x) = f(y) + \frac{\partial f}{\partial x'}(y^*)(x-y)$ with y^* some element of $\mathbb{R}^q$ that lies in the line that connects x and y (this line is well-defined since $\mathbb{R}^q$ is convex, and $\frac{\partial f}{\partial x'}(y^*)$ is well-defined since f is everywhere differentiable)

$$(*) \Leftrightarrow \quad \underbrace{f(x) - f(y)}_{\in \mathbb{R}^p} = \overbrace{\frac{\partial f}{\partial x'}(y^*)(x-y)}^{\in \mathbb{R}^p} \Rightarrow$$

Euclidean norm in $\mathbb{R}^q$

$$\|f(x) - f(y)\| = \|\underbrace{\frac{\partial f}{\partial x'}(y^*)}_{A}\underbrace{(x-y)}_{x}\| \underset{\text{submult.}}{\leq} \|\frac{\partial f}{\partial x'}(y^*)\|_F \|x-y\|$$

$$\leq \underline{\sup_{x\in\mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\| \|x-y\|}$$

Since $\sup_{x\in\mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\| < +\infty$ we obtain that for the well defined $C_f := \sup_{x\in\mathbb{R}^q} \|\frac{\partial f}{\partial x'}(x)\|$ (this is independent of x, y)

we obtain

$$\|f(x) - f(y)\| \leq C_f \|x-y\| \Leftrightarrow$$

$$\underset{\text{in } \mathbb{R}^p}{d_I}(f(x), f(y)) \leq C_f \underset{\text{in } \mathbb{R}^q}{d_I}(x, y)$$

and since x, y are arbitrary and C_f is independent of them we have proven that:

$$\forall x,y \in \mathbb{R}^q, \quad d_I(f(x), f(y)) \leq C_f \, d_I(x,y)$$

and the result follows. □

Remark. There is a partial converse to the previous theorem. If $f: \mathbb{R}^q \to \mathbb{R}^p$ is (d_I/d_I -) Lipschitz continuous then f is almost everywhere differentiable with bounded derivative (where defined). The term "almost" means that there can exist $x \in \mathbb{R}^q$ at which f is not differentiable that form a negligible subset of $\mathbb{R}^q$.

Remark. More generally the previous theorem holds whenever $f: A \to \mathbb{R}^p$, $A \subseteq \mathbb{R}^q$, f is everywhere differentiable in A with bounded derivative, and $\forall x,y \in A$ there exists a line in A that connects them (e.g. A is convex).

Example: $p = q = 1$, $d_I = d_u$, $A = (-1,1)$ $f: (-1,1) \to \mathbb{R}$

$f(x) = e^x$ (the exponential function restricted to $(-1,1)$).

convex

$$\frac{df}{dx} = e^x \;\forall x \in (-1,1), \quad \sup_{x \in (-1,1)} \left\| \frac{\partial f}{\partial x}(x) \right\|_I = \sup_{x \in (-1,1)} |e^x| = \sup_{x \in (-1,1)} e^x$$

$= e^1 = e < +\infty$. Hence the exponential function restricted to $(-1,1)$ is Lipschitz continuous with $C_f = e$.

Exe. Is $f(x) = e^x : \mathbb{R} \to \mathbb{R}$ Lipschitz continuous?

Compositional Property of Lipschitz Continuity.

(X, d_X), (Y, d_Y), (Z, d_Z). $X \xrightarrow{f} Y$ (d_Y/d_X -) Lipshitz continuous, $Y \xrightarrow{g} Z$ (d_Z/d_Y -) Lipshitz Continuous.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

d_X $\qquad$ d_Z

Is $g \circ f$ $\left(d_Z/d_X\text{-}\right)$ Lipschitz-Continuous?

Let $x, y \in X$, then $d_Z(g(\underbrace{f(x)}_{\in Y}), g(\underbrace{f(y)}_{\in Y})) \leq$ $\exists\, C_g > 0$ Lipsh. coef. of g

$C_g\, d_Y(f(x), f(y)) \leq C_g \cdot C_f\, d_X(x,y)$ $\exists\, C_f > 0$ Lipsh. Coef. of f

Since x, y are arbitrary we have that:

$$\forall x, y \in X \quad d_Z(g\circ f(x), g \circ f(y)) \leq C_g \cdot C_f\, d_X(x,y)$$

Hence $g \circ f$ is d_Z/d_X - Lipschitz continuous and $C_{g\circ f} = C_g \cdot C_f$.

(This easily extends to any finite composition).