Lecture 05-05-20

Reminder: Sequential convergence in metric spaces

- (X,d) a m.s., (x_n) a sequence of elements of X, x is a limit of (x_n) (w.r.t. d) iff $\forall \varepsilon>0$, $x_n \in O_d(x,\varepsilon)$ $\forall n \in \mathbb{N}$ except for a finite number of n (that may depend on ε).
- Equivalent if closed balls are used.

$(x_0, x_1, x_2, \dots, x_n, \dots)$

Uniqueness: If (x_n) has a limit (w.r.t. d) then it is unique.

Proof. Suppose that (x_n) is convergent. Suppose that its limit is not unique. This means that it has at least two limits, say $x,y \in X$ $(x \neq y)$. Since $x \neq y \Leftrightarrow d(x,y)>0$, hence due to the separation with open balls property in metric spaces, $\exists\, \varepsilon,\delta>0$: $O_d(x,\varepsilon) \cap O_d(y,\delta) = \emptyset$ $\left(\varepsilon=\delta=\frac{d(x,y)}{2}\right)$. Since $x_n \to x \Rightarrow x_n \in O_d(x,\varepsilon)$ $\forall n \in \mathbb{N}$ except for a finite number of n. Hence only a finite number of x_n's lie in $O_d^c(x,\varepsilon)$. But $O_d(y,\delta) \subseteq O_d^c(x,\varepsilon)$ hence only a finite number of x_n's can lie inside $O_d(y,\delta)$. This is impossible because $x_n \to y$ by assumption. $\square$

Remark: In pseudo-metric spaces uniqueness may fail.

Lemma (Boundness). If (x_n) is convergent (w.r.t. d) then (x_n) is bounded (w.r.t. d).

Reminder: (x_n) is considered bounded w.r.t. d iff $\exists y \in X$, $\delta > 0$: $x_n \in O_d(y, \delta)$ (equiv. $x_n \in O_d[y, \delta]$), $\forall n \in \mathbb{N}$.

Proof. We have that $x_n \to x$ for some $x \in X$ (we use closed balls without loss of generality). We know that $x_n \in O_d[x, 1]$ $\forall n \in \mathbb{N}$ except for a finite number of n.

Suppose that $x_{n_1}, x_{n_2}, \dots, x_{n_k} \notin O_d[x, 1]$ for some $k > 0$.

Since k is $\underline{\underline{\text{finite}}}$ & $\max\big(\underbrace{d(x, x_{n_1}), d(x, x_{n_2}), \dots, d(x, x_{n_k}), 1}_{k+1 \text{ reals}}\big) < +\infty$

Hence we can choose $\delta := \max(d(x, x_{n_1}), d(x, x_{n_2}), \dots, d(x, x_{n_k}), 1)$ as a new radius and consider $O_d[x, \delta] \supseteq O_d[x, 1]$ hence every element of the sequence that belong in $O_d[x, 1]$ also belong in $O_d[x, \delta]$, but also for $i = 1, \dots, k$ we have that $d(x, x_{n_i}) \leq \max(d(x, x_{n_1}), \dots, d(x, x_{n_k}), 1) = \delta$ hence $x_{n_i} \in O_d[x, \delta]$ $\forall i = 1, \dots, k$. Hence $x_n \in O_d[x, \delta]$ $\underline{\forall n \in \mathbb{N}}$.

Hence (x_n) is bounded. $\square$

Remark. The converse does not generally hold.

Lemma. Every (x_n) that is eventually constant (say at $x \in X$) is convergent (to c) w.r.t. every d. (universal property)

Proof. Since (x_n) is eventually constant (at $c \in X$)

(x_n) will have the form $(x_0, x_1, \dots, x_k, c, c, \dots, c, \dots)$

where $x_i \neq c$ possibly for some $i = 0, 1, \dots, k$, for some k.

Consider $O_d(c, \varepsilon)$ for any d, $\varepsilon > 0$. Then $x_n \in O_d(c, \varepsilon)$ $\forall n \in \mathbb{N}$ except perhaps for some $n = 0$, and/or 1, …, and/or k. Hence $x_n \to c$ $\forall d$. $\square$

Examples:

1. $X = \mathbb{R}$, $d = d_u$, $x_n = \frac{1}{n+1}$, $n \in \mathbb{N}$, $(1, 1/2, 1/3, \dots, 1/_{n+1}, \dots)$.

We have that $\frac{1}{n+1} \to 0$ w.r.t. d_u since for $\varepsilon > 0$ we have

$O_{d_u}(0, \varepsilon) = (-\varepsilon, \varepsilon) \ni \frac{1}{n+1}$ $\forall n >$ smallest natural greater than or equal $1/\varepsilon - 1$.

2. X arbitrary, $d = d_d$ (hence we are examining the issue of sequential convergence in discrete spaces). Remember that $\forall x \in X$, $\varepsilon > 0$, $O_{d_d}(x, \varepsilon) = \begin{cases} X, & \varepsilon > 1 \\ \{x\}, & \varepsilon \leq 1 \end{cases}$. Hence in order for (x_n) (with $x_n \in X$ $\forall n \in \mathbb{N}$) to converge to x, x_n must equal x $\forall n \in \mathbb{N}$ except for a finite number of n. Hence in order for (x_n) to be convergent w.r.t. d_d it has to be eventually

constant. Hence combining this with the previous Universal property we have proven that:

In a discrete space a sequence is convergent iff it is eventually constant.

This showed us:

- The issue of convergence or divergence of a sequence depends among others on d (e.g. $(1/_{n+1})$ in $\mathbb{R}$ converges to 0 w.r.t. d_u, yet diverges a non-eventually constant of).
- Convergence in discrete spaces seems trivial!

3. $X = B(Y, \mathbb{R})$, $d = d_{sup}$, if (f_n) is a sequence in X (i.e. $f_n \in B(Y,\mathbb{R})$ $\forall n \in \mathbb{N}$), and it is convergent w.r.t. d_{sup} to some limit $f \in B(Y, \mathbb{R})$ - we say that f_n is uniformly convergent to f.

Compare the above with: if $x \in Y$, then $(f_n(x)) := (f_0(x), f_1(x), \dots, f_n(x), \dots)$ is a sequence of real numbers. We can ask for which $x \in Y$ the real sequence $(f_n(x))$ converges w.r.t. d_u.

We can thus define another concept of convergence for (f_n).

We say that $f: Y^* \to \mathbb{R}$ (Y^* is a suitable subset of Y - see below) is the pointwise limit of (f_n) iff $\forall x \in Y^*$, the real sequence $(f_n(x))$ is convergent w.r.t. d_u, and $f(x) = \lim f_n(x)$ (w.r.t. d).

[the pointwise limit exists iff $Y^* \neq \emptyset$ i.e. iff there exists at least a $x \in Y$ for which the real sequence $(f_n(x))$ is convergent w.r.t. d_u].

What is the relation between pointwise and uniform convergence:

Pointwise convergence of (f_n) to f is defined by the convergence of $|f_n(x) - f(x)| \to 0 \quad \forall x \in Y^*$

Uniform convergence of (f_n) to f is defined by the convergence of $d_{sup}(f_n, f) = \sup_{x \in Y} |f_n(x) - f(x)| \to 0$

We have that $\forall x \in X \quad |f_n(x) - f(x)| \leq \overbrace{\sup_{x \in Y} |f_n(x) - f(x)|}^{\to d_{sup}}$

Hence if $\sup_{x \in Y} |f_n(x) - f(x)| \xrightarrow{d_u \ \forall x} 0 \Rightarrow |f_n(x) - f(x)| \to 0 \quad \forall x \in Y$

Hence if (f_n) converges uniformly to f the necessarily it converges to the same limit pointwisely.

Does the converse hold? The following counterexample shows that uniform convergence is genuinely stronger than pointwise convergence:

$Y = [0,1]$, $X = B([0,1], \mathbb{R})$, and $f_n(x) := x^n$, $x \in [0,1]$

(hence we are considering the following sequence in

$$(\underbrace{x^0}_{\substack{\parallel \\ \text{Constant} \\ \text{at } 1}}, x, x^2, \dots, x^n, \dots))$$

Is (f_n) pointwise convergent?

$$f_n(x) = x^n \xrightarrow[n\to+\infty]{} f(x) := \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases} \in B([0,1], \mathbb{R})$$

Pointwisely

Does (f_n) converge uniformly to f? We have that

$$d_{sup}(f_n, f) = \sup_{x\in[0,1]} |f_n(x) - f(x)| =$$

$$= \sup_{x\in[0,1]} |x^n - f(x)| = 1 \underset{n\to+\infty}{\not\to} 0$$

$$\sup_{x\in[0,1)} |x^n|$$

hence f_n does not converge to f uniformly (this directly informs us also that (f_n) does not have a

Uniform limit) - hence this also constitutes an example of a divergent sequence.

$\square$

Sequential Convergence and Metrics Comparison

$X \neq \emptyset$, d_1, d_2 are metrics definable on X,

$\exists c > 0 :\quad d_1 \leq c d_2.$

Let (x_n), $(x_n \in X\ n \in \mathbb{N})$ and $x_n \to x$ w.r.t. d_2.

Hence $d_2(x_n, x) \to 0$ as $n \to +\infty$. We know that

for all $n \in \mathbb{N}$ $\quad \underbrace{d_1(x_n, x)}_{\to 0 \text{ as } n \to +\infty \text{ since } d_1 \geq 0.} \leq c\, \overbrace{d_2(x_n, x)}^{\to 0}$

Hence if $x_n \to x$ w.r.t. d_2 then $x_n \to x$ w.r.t. d_1

$\square$