Lecture 28/04/20

- If (X,d) a metric space, $A \subseteq X$, $\varepsilon > 0$

  $N(\varepsilon, d, A) :=$ the "smallest" number of closed balls of ε radius that covers A. [Covering number of A corresponding to ε].

- Variations of the above are definable if we use open balls and/or if we restrict the centers to lie in A. Those variations will not necessarily equal each other but it is possible to prove that they have equivalent asymptotic behavior as $\varepsilon \downarrow 0$, hence they convey the same information on the complexity of A.
- A is t.b. iff $N(\varepsilon, d, A) \in \mathbb{R}$ $\forall \varepsilon > 0$.
- $\ln N(\varepsilon, d, A)$ is termed as metric entropy number of A.

E.g. $X = \mathbb{R}$, $d = d_u$, $A = [\alpha, \beta] = O_{d_u}[\frac{\alpha+\beta}{2}, \frac{\beta-\alpha}{2}]$
$\alpha, \beta \in \mathbb{R}$

we have easily shown $N(\varepsilon, d_u, [\alpha,\beta]) \sim \frac{c}{\varepsilon}$ depends on α, β

Since $\frac{c}{\varepsilon} \in \mathbb{R}$ $\forall \varepsilon > 0$ this essentially showed us that boundness is equivalent to total boundness in $\mathbb{R}, d_u$

- Metric Entropy: $\ln N(\varepsilon, d_u, [\alpha,\beta]) \sim \ln c - \ln \varepsilon \sim -\ln \varepsilon$ dominant term as $\varepsilon \downarrow 0$

E.g. $X = \mathbb{R}^d$ $d = d_I$ it is possible to prove that:
$\forall x \in \mathbb{R}^d, \delta > 0$, $A = O_{d_I}[x, \delta]$, $N(\varepsilon, d_I, O_{d_I}[x,\delta]) \sim \frac{c}{\varepsilon^d}$
for some suitable $c > 0$. Since $\frac{c}{\varepsilon^d} \in \mathbb{R}$ $\forall \varepsilon > 0$ every closed ball in this metric space is totally bounded.

Hence boundness is equivalent to total boundness in $\mathbb{R}^d, d_I$.

– Metric Entropy: $\ln N(\varepsilon, d_I, O_{d_I}[x,\delta]) \sim -d\ln\varepsilon$

E.g. $X = B([0,1], \mathbb{R})$, $d = d_{sup}$, $A = \{f : [0,1] \to \mathbb{R}, f(0) = 0,$
$L > 0$ $\qquad |f(x) - f(y)| \leq L|x-y|\}$

We know that A is bounded (uniform boundness) but is it totally bounded?

We can show that $N(\varepsilon, d_{sup}, A) \sim \exp(C/\varepsilon)$ and C is a positive constant independent of ε that depends on L.

Since $\exp(C/\varepsilon)$ is finite $\forall \varepsilon > 0$ we conclude that A is totally bounded w.r.t. d_{sup}. By comparing the asymptotic behavior of $e^{C/\varepsilon}$ with C/ε^d we can conclude that the particular A is "more complex" than any Euclidean ball

Metric Comparison and total boundness

X, d_1, d_2 well defined metrics, for some $c > 0$, $d_1 \leq c d_2$.

$\Longleftrightarrow \forall x \in X, \varepsilon > 0 \quad O_{d_2}(x, \varepsilon) \subseteq O_{d_1}(x, c\varepsilon) \Longleftrightarrow \forall x \in X, \forall \delta > 0$

$\delta = c\varepsilon$ bijection since $c > 0$.

$(*)$ $O_{d_2}(x, \delta/c) \subseteq O_{d_1}(x, \delta)$

Suppose that A is t.b. w.r.t. d_2. In order to see whether A is t.b. w.r.t. d_1, let $\delta > 0$, since A is d_2 t.b. we have that there exists a finite cover of open balls or radius δ/c of A w.r.t. d_2, hence due to $(*)$ there exists a finite cover of open balls of radius δ of A w.r.t. d_1 (Just use the centers of the previous cover). Hence since δ is arbitrary A is t.b. w.r.t. d_1. Hence we have proven:

Lemma. Total boundness w.r.t. d_2 $\Rightarrow$ Total boundness w.r.t. d_1.

Furthermore due to $(*)$ we have that

$$N(\delta/c, d_2, A) \geq N(\delta, d_1, A).$$

Lemma. Suppose that for $c_1, c_2 > 0$: $c_1 d_2 \leq d_1 \leq c_2 d_2$. $(**)$

Then Total boundness w.r.t. d_1 $\Leftrightarrow$ Total boundness w.r.t. d_2.

For example in $\mathbb{R}^d$ boundness is equivalent to total boundness w.r.t. any of the metrics d_I, d_{max}, d_{II}.

(Exercise: provide the details)

Exercise: what is the relation between the respective covering numbers when $(**)$ holds?

– Total boundness can be very useful in applications related with the issue of convergence in function spaces.

Topological Notions on Metric Spaces:

Convergence – Continuity

- A topology on a set X is a "specification" of which subsets of X are considered open and dually closed.
- Topologies facilitate the examination of notions of convergence (limits) and continuity.
- In metric spaces balls "produce" topologies.

We will try to largely avoid the constructions implied above and examine convergence and continuity by solely restricting ourselves to the use of open (closed) balls.

Convergence. (of sequences in metric spaces)

Preparation: Let X be a non-empty set. A sequence of elements of X (X-valued sequence) is a function $\mathbb{N} \longrightarrow X$ or equivalently a vector of elements of X that has an initial element, does not have a final element and has as many elements as $\mathbb{N}$.

E.g. $X = \mathbb{R}$, $f : \mathbb{N} \to \mathbb{R}$, $f(n) = \frac{1}{n+1}$, or equivalently it is the vector $(1, 1/2, 1/3, \dots, 1/{n+1}, \dots)$

this is an example of a real sequence

E.g. $X = \mathcal{B}([0,1], \mathbb{R})$, and consider the sequence

$$(1, x, x^2, \dots, x^{n+1}, \dots), \quad x \in [0,1]$$

- A property P is said to hold almost everywhere for a given sequence iff every element of the sequence obeys P except for a finite number of elements.

E.g. A X-valued sequence is termed almost everywhere (or eventually) constant iff for some $c \in X$ we have that $x_n = c$ $\forall n \in \mathbb{N}$ except for a finite number of n's

is the $n+1^{th}$ element of the sequence $(x_0, x_1, x_2, \dots, x_n, \dots)$

This is equivalent to that $\exists n^* \in \mathbb{N}$:

$$(x_0, x_1, x_2, \dots, x_{n^*-1}, c, c, \dots, c, \dots)$$

Consider now a metric space X, d and $(x_0, x_1, \dots, x_n, \dots)$ is a X-valued sequence.

Definition. The sequence above is convergent (w.r.t. d) iff $\exists x \in X$ w.r.t. which we have that:

$\forall \varepsilon > 0$ the sequence lies almost everywhere in $O_d(x, \varepsilon)$ (i.e. it is allowed that a finite part of the sequence can lie outside $O_d(x, \varepsilon)$)

x is termed limit of the sequence (w.r.t. d) and it is denoted with $\lim x_n$ or $d\text{-}\lim x_n$ etc.

(the convergence is also usually denoted with $x_n \to x$)

Remark: the definition above allows that the elements and their number of the sequence that lie outside $O_d(x,\varepsilon)$ may depend on ε.

Remark: Suppose that $x_n \to x$ (w.r.t. d), i.e. due to the definition above $\forall \varepsilon > 0$, almost every element of the sequence lies in $O_d(x,\varepsilon) \Leftrightarrow \underline{d(x_n,x) < \varepsilon}$ for almost every n (i.e. this holds $\forall n \in \mathbb{N}$ except for a finite number of n's). Now consider the vector

$$\left(\underline{d(x_0,x), d(x_1,x), d(x_2,x), \dots, d(x_n,x), \dots}\right)$$

this is a $\mathbb{R}$-valued sequence, and notice that

$$d(x_n,x) < \varepsilon \Leftrightarrow |d(x_n,x)| < \varepsilon \Leftrightarrow |d(x_n,x) - 0| < \varepsilon$$

$\Leftrightarrow d(x_n,x) \in O_{d_1}(0,\varepsilon)$, and since for any $\varepsilon > 0$ this holds $\forall n \in \mathbb{N}$ except for a finite number of n's the previous is equivalent to that $\lim_{n\to+\infty} d(x_n,x) = 0$.

Hence $x_n \to x \Leftrightarrow d(x_n,x) \to 0$. □

Remark. In the definition above we can equivalently replace open balls with the respective closed balls.

This is due to the following: (def. based on open balls $\Rightarrow$ def. based on closed balls)

If $x_n \to x$ via the original definition then $\forall \varepsilon > 0$ almost every element of the sequence lies $O_d(x,\varepsilon)$ but since $O_d(x,\varepsilon) \subseteq O_d[x,\varepsilon]$ then almost every element of the sequence lies in $O_d[x,\varepsilon]$.

(def. based on closed balls $\Rightarrow$ def. based on open balls)

If $x_n \to x$ via the "closed-balls" based definition. Let $\varepsilon > 0$. Due to the closed balls definition almost every element of the sequence lies $O_d[x,\varepsilon/2]$. But we have that $O_d[x,\varepsilon/2] \subseteq O_d(x,\varepsilon)$ and thereby almost every element of the sequence lie in $O_d(x,\varepsilon)$. The result follows since ε is arbitrary.

Lemma. If the sequence is convergent the its limit is unique.

Proof.