Lecture 07/04/20

Further details on boundness

1. X, d_1, d_2, $A \subseteq X$, $d_1 \leq c d_2$, $c > 0$.

If A is bounded w.r.t. d_2 then A is also bounded w.r.t. d_1.

Corollary. Suppose that $\exists\, C_1, C_2 > 0$: $C_1 d_2 \leq d_1 \leq C_2 d_2$ (*)

$\Leftrightarrow$ (– $d_1 \leq C^* d_2$ for
– $d_2 \leq C_* d_1$ $C^*, C_* > 0$)

then A is bounded w.r.t. d_1 iff
A is bounded w.r.t. d_2

(i.e. d_1, d_2 completely "agree" w.r.t. boundness)

E.g. $X = \mathbb{R}^d$, d_I, d_{max}, d_{11} and we know that every possible pair of them satisfies a relation of the previous type on. Hence d_I, d_{max}, d_{11} completely "agree" w.r.t. boundness on $\mathbb{R}^d$.

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2. Boundness and Metric Subspaces

Remember: i. (X,d) metric space, $\emptyset \neq A \subseteq X$
we can "restrict" d to A (obtaining d_A) hence we obtain (A, d_A) as a metric subspace of the previous.

ii. $x \in A$, $\varepsilon > 0$, $O_{d_A}(x,\varepsilon) = O_d(x,\varepsilon) \cap A$
(something similar holds for closed balls)

iii. A is bounded iff $\exists\, x \in A, \varepsilon > 0$:
$O_d(x,\varepsilon) \supseteq A$.

Lemma. (A, d_A) is bounded iff A is bounded w.r.t. d.

(If true it means that we may only consider the cases of whether a metric (sub-) space if bounded.)

Proof.

α. if A, d_A is bounded then A is a bounded subset of X.

β. if A is a bounded subset of X then A, d_A is bounded.

α. we know that A, d_A is bounded $\Leftrightarrow \exists\, x \in A$, $\varepsilon > 0$:
$O_{d_A}(x,\varepsilon) \supseteq A$, but we have that $O_{d_A}(x,\varepsilon) = O_d(x,\varepsilon) \cap A$ $\Rightarrow$ $O_d(x,\varepsilon) \supseteq A \Leftrightarrow A$ is a bounded subset of X.

β. we know that A is a bounded subset of X $\Leftrightarrow$
$\exists\, x \in A, \varepsilon > 0$: $O_d(x,\varepsilon) \supseteq A$. Consider $O_{d_A}(x,\varepsilon) :=$
$O_d(x,\varepsilon) \cap A \supseteq A$. Hence A, d_A is bounded. $\square$

3. "Fragility of boundness w.r.t. limit"

Consider $X=\mathbb{R}$, for $n=1,2,3,\ldots$ Consider the
$\in \mathbb{N}^*$

following $d_n : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, defined by:

$$x,y\in\mathbb{R},\quad d_n(x,y)=\begin{cases} |x-y|, & |x-y|<n \\ n, & |x-y|\geq n \end{cases}$$

$$= \min\left(d_u(x,y), n\right) = \begin{cases} d_u(x,y), & d_u(x,y)<n \\ n, & d_u(x,y)\geq n \end{cases}$$

Exer. Show that d_n is a well defined metric for any $n=1,2,\ldots$

(If $n=0$ was allowed, d_0 would be a pseudo-metric)

Hence we have essentially constructed a "sequence" of metric spaces, $(\mathbb{R},d_1)$, $(\mathbb{R},d_2)$, $(\mathbb{R},d_3)$, $\ldots$, $(\mathbb{R},d_n)$, $\ldots$

bounded

a. Consider $\mathbb{R}, d_n$, let $x\in\mathbb{R}$, and consider furthermore

$$O_{d_n}(x,n+1)=\{y\in\mathbb{R}: d_n(x,y)<n+1\} = \mathbb{R}.$$

Hence $\mathbb{R}$ is bounded w.r.t. d_n $\forall n=1,2,\ldots$

b. What happens to $d_n(x,y)$ as $n\to+\infty$?
fixed

We have that eventually (for large enough n)

$d_n(x,y) = |x-y|$, hence $\lim_{n\to+\infty} d_n(x,y) = |x-y| = d_u(x,y)$

Since x,y were arbitrary we have that

$$\forall x,y \in \mathbb{R} \quad d_n(x,y) \xrightarrow[n\to+\infty]{} d_u(x,y)$$

(in some way - compare with the notion of pointwise convergence - d_n converges to d_u)

Hence in some way $\mathbb{R}, d_u$ is some kind of limit of the previous sequence of metric space.

c. $\mathbb{R}, d_u$ is <u>not</u> bounded

Hence we have an example for which boundness is "lost in the limit".

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4. We have in many cases worked with $B(X,\mathbb{R})$, d_{sup}

$\{f : X \to \mathbb{R}, \text{ bounded}\}$

- f is bounded $\Leftrightarrow$ $f(X)$ is a bounded subset of $\mathbb{R}$ w.r.t d_u.
- $d_{sup}(f,g) = \sup_{x\in X} |f(x)-g(x)| = \sup_{x\in X} d_u(f(x),g(x))$.

We can now easily generalize the previous construction

Suppose that we have a metric space (X,d)

- $f : Y \to X$ will be considered bounded w.r.t. d iff $f(Y)$ is a bounded subset of X w.r.t. d i.e. $\exists z \in X, \varepsilon > 0 : O_d(z, \varepsilon) \supseteq f(Y)$, hence $B(Y, X)$ is well defined. (always $\neq \emptyset$ - constant functions)

- We can analogousle define the analogue of d_{sup} in this extended setting:

$$f, g \in B(Y, X), \quad d^*_{sup}(f, g) = \sup_{x \in Y} d(f(x), g(x))$$

Exe. Show that d^*_{sup} is a well-defined metric.

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5. (X, d), $\varepsilon > 0$, $Z \subseteq X$. Consider the following collection of open balls:

$$\mathcal{O}(Z, \varepsilon) = \{ O_d(x, \varepsilon), \ \forall x \in Z \}$$

The relevant collection of closed balls is

$$\bar{\mathcal{O}}(Z, \varepsilon) = \{ O_d[x, \varepsilon], \ \forall x \in Z \}$$

When Z is finite then the collections above are termed finite.

When $A \subseteq X$, we say that A is covered by $\mathcal{O}(Z, \varepsilon)$ iff $\bigcup_{x \in Z} O_d(x, \varepsilon) \supseteq A$

(analogously for $\bar{\mathcal{O}}(Z, \varepsilon)$).

Theorem. A is bounded w.r.t. d iff $\exists\, \varepsilon > 0$ w.r.t. which there exists a finite $\mathcal{O}(Z, \varepsilon)$ that covers A.

[we can equivalently use closed balls – Exe!]

Proof.

α. Suppose that A is bounded $\Leftrightarrow \exists x, \varepsilon > 0$: $O_d(x, \varepsilon) \supseteq A$. Hence $\mathcal{O}(Z, \varepsilon) = \{O_d(x, \varepsilon)\}$, $Z = \{x\}$.

b. Suppose that there exists $\varepsilon > 0$ w.r.t. there exists a finite $\mathcal{O}(Z, \varepsilon)$ that covers A.

$\mathcal{O}(Z, \varepsilon)$ has the form $\{O_d(x_1, \varepsilon), O_d(x_2, \varepsilon), \ldots, O_d(x_n, \varepsilon)\}$ for some $n \in \mathbb{N}^*$. $(Z = \{x_1, x_2, \ldots, x_n\})$

We will try to construct a "large enough" ball that covers $\bigcup_{i=1}^{n} O_d(x_i, \varepsilon)$. Without loss of generality choose x_1 as center.

Let $\delta := \max_{i=2,\dots,n} d(x_1, x_i) + \varepsilon + 1 > 0$

$n \in \mathbb{N}^* \Rightarrow$ this is well defined

We have that $O_d(x_1, \delta) \supseteq \bigcup_{i=1}^{n} O_d(x_i, \varepsilon)$, this is due to that: let $y \in \bigcup_{i=1}^{n} O_d(x_i, \varepsilon) \Rightarrow y \in O_d(x_i, \varepsilon)$ for some i, that is $d(x_i, y) < \varepsilon$ for some i.

We have that $d(x_1, y) \leq d(x_1, x_i) + d(x_i, y)^{< \varepsilon}$

$$< \max_{i=2,\dots,n} d(x_1, x_i) + \varepsilon$$

$\leq \max_{i=2,\dots,n} d(x_1, x_i)$

$$< \max_{i=2,\dots,n} d(x_1, x_i) + \varepsilon + 1 = \delta$$

Hence $d(x_1, y) < \delta \iff y \in O_d(x_1, \delta) \Rightarrow$

$$O_d(x_1, \delta) \supseteq \bigcup_{i=1}^{n} O_d(x_i, \varepsilon) \supseteq A$$

hence $A \subseteq O_d(x_1, \delta)$ hence A is bounded. $\square$

Total Boundness

Definition. A is totally bounded (w.r.t. d) iff $\forall \varepsilon > 0$ $\exists Z$ finite such that the collection $O(Z, \varepsilon)$ covers A.

[Dually we can consider collections of closed balls].

Remark. The definition allows the dependence of Z on ε. Hence both the centers and the cardinality of Z is allowed to change with ε.

Remark. When A is totally bounded then $\forall \varepsilon>0$ Z can be chosen as a subset of A (i.e. the ball centers can be chosen to lie inside A).

Proof. Suppose that A is totally bounded. Let $\varepsilon>0$. Due to the previous for $\delta=\varepsilon/2$ there exists a finite Z such that the collection $\mathcal{O}(Z,\varepsilon/2)$ covers A. The collection $\mathcal{O}(Z,\varepsilon/2)$ is the set of open balls $\{O_d(x_1,\varepsilon/2), O_d(x_2,\varepsilon/2), \dots, O_d(x_n,\varepsilon/2)\}$ ($Z=\{x_1,x_2,\dots,x_n\}$ for n that may depend on $\varepsilon/2$).

If $Z\subseteq A$ then the collection $\mathcal{O}(Z,\varepsilon)$ covers A since $O_d(x_i,\varepsilon/2)\subseteq O_d(x_i,\varepsilon)$ $\forall i=1,\dots,n$. If $Z\nsubseteq A$ then:

Suppose that $x_1\notin A$ without loss of generality.

Since $\bigcup_{i=1}^{n} O_d(x_i, \varepsilon/2) \supseteq A$ we consider

$O_d(x_1, \varepsilon/2) \cap A$. If this is empty then the ball

$O_d(x_1, \varepsilon/2)$ is not needed in the covering and thereby

x_1 can be discarded from Z and we can move on

to the next element of Z that does not lie in A.

So suppose that $O_d(x_1, \varepsilon/2) \cap A$ is not empty.

Let $y \in O_d(x_1, \varepsilon/2)$. Consider $O_d(y, \varepsilon)$. We have

that if $z \in O_d(x_1, \varepsilon/2) \cap A$ then $z \in O_d(y, \varepsilon)$ ...

We will continue it next time.