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Study of some purely Topological Notions

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Basic Calculus and Real Analysis have spelled out for us what is perhaps our natural intuition of distance is between two real numbers or more generally elements of $\mathbb{R}^N$ with $N \in \mathbb{N}^*$. Let $x, y \in \mathbb{R}$, then we usually say that their distance is simply their absolute difference, $|x - y|$. Let $x, y \in \mathbb{R}^2$, then their distance is usually defined using the Pythagorean Theorem, $\sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Study of metric spaces has shown us that we can generalize the idea of the distance between elements of $\mathbb{R}$ or $\mathbb{R}^N$. We realize that the above mentioned absolute distance or the Euclidean distance are just specific cases of pairings of $\mathbb{R}$ and $\mathbb{R}^N$ with specific metric functions and that we could have used other appropriate functions to define the notion of distance between their elements. We also realize that we can define distance for different kinds of sets, possibly much more "exotic" than the set of real numbers, by pairing them with appropriate metric functions which satisfy a list of desired properties on the set in question. We call such pairings *metric spaces* and they are comprised of a non-empty *carrier set* and a well behaved *metric function*.

General Topology abstracts away with the idea of a numerically identifiable distance and replaces *metric functions* with so called *topologies* on the carrier set. This pairing is now called a *topological space*.

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Openness (and closedness) in metric spaces

Definition

Let (X, d) be a metric space and A a subset of X (not necessarily non-empty). A is a d-open subset of X (i.e. open with respect to the metric d) iff

$$\forall x \in A,\, \exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

The x subscript on ε_x means that generally this radius may depend on the element in question.

Notice that the carrier set is always a d-open subset of itself, because for every one of its elements we can find a radius such that the constructed d-open ball is a subset of the carrier set (and in fact any radius will do).

We also axiomatically say that the empty set is a d-open subset of the carrier set of any metric space (but also because it can be shown that the carrier set is d-closed via sequential d-convergence).

Finally, observe that d-open balls are also d-open subsets of their carrier sets.

Lemma

Let (X, d) be a metric space. Any d-open ball in it is a d-open subset of X.

Proof

Choose an arbitrary $x \in X$ and radius $\varepsilon > 0$ and define the d-open ball $\mathcal{O}_d(x, \varepsilon)$.
For some $y \in \mathcal{O}_d(x, \varepsilon)$ define $\delta \coloneqq \varepsilon - d(x, y) > 0$.
Then $\mathcal{O}_d(y, \delta) \subseteq \mathcal{O}_d(x, \varepsilon)$ (to see why see Problem Set 2 Exercise 3).
So $\mathcal{O}_d(x, \varepsilon)$ is a d-open of subset of X. Since x and ε were chosen arbitrarily, every d-open ball is a d-open of subset of X.

Furthermore, the following properties hold with respect to d-openness:

- Arbitrary unions of d-open sets are d-open.

- Finite intersections of d-open sets are d-open.

  To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric (absolute difference).

  Let A_n be subsets of $\mathbb{R}$ such that

  $$A_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \forall\, n \in \mathbb{N}^*$$

  With respect to the usual metric on $\mathbb{R}$, all A_n are d-open subsets of $\mathbb{R}$.

Now notice that there is an infinite number of such sets and their only common element is 0. So their intersection (an infinite intersection of d-open sets) is $\{0\}$.

But also notice that $\forall \varepsilon > 0 \; \mathcal{O}_d(0,\varepsilon) \supset \{0\}$. So $\nexists \; \varepsilon > 0 : \mathcal{O}_d(0,\varepsilon) \subseteq \{0\}$.

Thus we have found an infinite intersection of d-open sets that is not d-open.

Definition

Let (X, d) be a metric space. Any subset, A, of X is termed *a d-closed subset of X* if its complement, A' (i.e. $A' = X \setminus A$), is a d-open subset of X.

This definition has a few implications.

First, the carrier set is a d-closed subset of itself, since the empty set is a d-open subset of X.

Secondly, the empty set is a d-closed subset of the carrier set, since the carrier set is a d-open subset of itself.

Finally, d-closed balls are d-closed subsets of their carrier sets.

Lemma

Let (X, d) be a metric space. Any d-closed ball in it is a d-closed subset of X.

Proof

Consider an arbitrary d-closed ball in X, $\mathcal{O}_d[x,\varepsilon]$. It suffices to show that its complement, $\mathcal{O}'_d[x,\varepsilon]$, is a d-open subset of X.

Let $y \in \mathcal{O}'_d[x,\varepsilon] \iff d(x,y) > \varepsilon$. We want to find a radius, δ, such that

$$\mathcal{O}_d(y,\delta) \subseteq \mathcal{O}'_d[x,\varepsilon]$$

$$\mathcal{O}_d(y,\delta) \bigcap \mathcal{O}_d[x,\varepsilon] = \varnothing$$

$$d(x,z) > \varepsilon, \; \forall \; z \in \mathcal{O}_d(y,\delta)$$

Now, define $\delta := d(x,y) - \varepsilon > 0$ and let $z \in \mathcal{O}_d(y,\delta)$. Naturally, the dual of the triangle inequality holds between x, y, and z (see Problem Set 1 Exercise 5)

$$d(x,z) \geq |d(x,y) - d(y,z)|$$

$$d(x,z) \geq |\varepsilon + \delta - d(y,z)|$$

$$d(x,z) \geq \varepsilon + |\delta - d(y,z)|$$

$$d(x,z) > \varepsilon$$

Since z was chosen arbitrarily, the above holds for all $z \in \mathcal{O}_d(y, \delta)$. So there exists a δ for y such that $d(x, z) > \varepsilon$, $\forall z \in \mathcal{O}_d(y, \delta)$.
Since y was chosen arbitrarily, the above holds for all $y \in \mathcal{O}'_d[x, \varepsilon]$. So $\mathcal{O}'_d[x, \varepsilon]$ is a d-open subset of X and $\mathcal{O}_d[x, \varepsilon]$ is d-closed.
Since $\mathcal{O}_d[x, \varepsilon]$ was chosen arbitrarily, the above holds for every d-closed ball.

Furthermore, the following properties hold with respect to d-closedness:

- Finite unions of d-closed sets are d-closed.

  To see why, consider this counter-example in $\mathbb{R}$ endowed with the usual metric (absolute difference).

  Let A_n be subsets of $\mathbb{R}$ such that

  $$A_n = \left[\frac{1}{n}, 1 - \frac{1}{n}\right], \forall\, n \in \mathbb{N}^*$$

  With respect to the usual metric on $\mathbb{R}$, all A_n are d-closed subsets of $\mathbb{R}$.

  Now notice that there is an infinite number of such sets and that

  $$\bigcup_{n=2}^{\infty} A_n = (0, 1)$$

  So their union (an infinite union of d-closed sets) is $B := (0, 1)$.

  But also notice that $B' = (-\infty, 0] \cup [1, +\infty)$ and that for its elements 0 and 1 there do not exist a radii $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that $\mathcal{O}_d(0, \varepsilon_0) \subseteq B'$ and $\mathcal{O}_d(1, \varepsilon_1) \subseteq B'$, respectively. So B' is not an open subset of $\mathbb{R}$ with respect to the usual metric and its complement B, an infinite union of closed sets, is not closed.

  Thus we have found an infinite union of d-closed sets that is not d-closed.

- Arbitrary intersections of d-closed sets are d-closed.

Topologies

Definition

A topology, τ, on a set, X, is a collection of subsets of X that satisfy the following properties:

1. $\varnothing$ and X belong to τ $(\varnothing, X \in \tau)$

2. any *arbitrary* union of elements of τ is also an element of τ (τ is closed with respect to arbitrary unions) $(A_i \in \tau, \forall\, i \in \mathcal{I} \Rightarrow \bigcup_{i\in\mathcal{I}} A_i \in \tau)$

3. any *finite* intersection of elements of τ is also an element of τ (τ is closed with respect to finite intersections) $(A_i \in \tau, \forall\, i \in \mathcal{I} \text{ finite } \Rightarrow \bigcap_{i\in\mathcal{I}} A_i \in \tau)$

The pair (X, τ) is termed *topological space.*

Any subset, A, of the carrier set, X, that also belongs to the chosen topology, τ, is by definition an open subset of X (and its complement, A', closed). A more vague definition of a topology could be that it is *a collection of "open" subsets of* X. Notice that we are no longer talking about d-openness, because openness is no longer determined by a metric function.

Lemma

For every metric function, d, that a non-empty set, X, is endowable with, there exists an implied topology, τ_d, where

$$\tau_d = \{A \subseteq X : A \text{ is } d\text{-open}\}$$

Proof

For τ_d to be a valid topology on X, it has to satisfy the properties 1-3 of topologies. The definition of τ_d basically says that every d-open subset of X belongs to it. So

1. $\varnothing$ and X belong to τ_d because they are d-open subsets of X.

2. Since all elements of τ_d are d-open subsets of X and arbitrary unions of d-open sets are d-open, then arbitrary unions of elements of τ_d also belong to τ_d.

3. Since all elements of τ_d are d-open subsets of X and finite intersections of d-open sets are d-open, then finite intersections of elements of τ_d also belong to τ_d.

Thus, τ_d is a topology on X.

An example of a topology generated by a metric is the discrete topology, τ_δ, which is generated by the discrete metric ($d_\delta(x, y) = 0$ if $x = y$, $d_\delta(x, y) = 1$ if $x \neq y$).

Let (X, d_δ) be a metric space and A an arbitrary subset of X. Then for any element, x, in A there exists a positive radius that is less than one, $0 < \varepsilon < 1$, such that $\mathcal{O}_{d_\delta}(x, \varepsilon) = \{x\} \subseteq A$. So A is d_δ-open and since it was chosen arbitrarily, every subset of X is open with respect to the discrete metric.

So the discrete topology of a set is its powerset, $\tau_\delta = P(X)$.

However, not all topologies can be generated by a metric. One such example can be the indiscrete topology, $\tau_I = \{\varnothing, X\}$ on a non singleton set. Let $X = \{a, b\}$ with $a \neq b$, and assume that there exists a metric d that we can endow X with and that it can produce the indiscrete topology, τ_I. Define $\varepsilon := d(a, b)$ and the d-open ball $\mathcal{O}_d(a, \varepsilon)$.

Notice that $\mathcal{O}_d(a, \varepsilon) \in \tau_I$, since it is a d-open subset of X and τ_I is the collection of d-open subsets of X (since it is generated by d).

Also, $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$ since open balls always contain their center.

Finally, $\mathcal{O}_d(a, \varepsilon) \neq X$ since it does not include b.

Thus, $\mathcal{O}_d(a, \varepsilon) \in \tau_I = \{\varnothing, X\}$ and $\mathcal{O}_d(a, \varepsilon) \neq \varnothing$, $\mathcal{O}_d(a, \varepsilon) \neq X$. Contradiction!

So τ_I cannot always be generated by a metric. We can, thus, say that *not all topological spaces are metrizable.*

Finally, not all collections of subsets constitute topologies. To see this, let $X = \{a, b, c\}$ and a collection of subsets, $\tau = \{\varnothing, \{a, b\}, \{b, c\}, X\}$. Then the intersection between $\{a, b\}$ and $\{b, c\}$ is $\{b\} \notin \tau$. So τ violates property 3 and is not a topology on X.

Definition

Let (X, τ) be a topological space and $x \in X$, then

$$\tau(x) = \{A \in \tau, x \in A\}$$

is called a *neighbouring system of x.*

Topological Convergence and Continuity in Metric Spaces

Definition

Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ a sequence in X. Then we say that $x \in X$ is a d-limit of $(x_n)_{n\in\mathbb{N}}$ iff

$$\forall\, \varepsilon > 0,\, \exists\, n_\varepsilon \in \mathbb{N} : \forall n \geq n_\varepsilon \quad x_n \in \mathcal{O}_d(x, \varepsilon)$$

and can denote $x = d - \lim(x_n)$ and equivalently write $x_n \to x$ (x_n tends to x).

Lemma

Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X, and $x \in X$. Consider τ_d as the topology on X generated by d. Then $x_n \to x$ iff $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A.

Proof

Suppose that x is the d-limit of $(x_n)_{n\in\mathbb{N}}$.

Choose an arbitrary $A \in \tau_d(x) \subseteq \tau_d$. By definition A is a d-open subset of X and $x \in A$. So

$$\exists\, \varepsilon_x > 0 : \mathcal{O}_d(x, \varepsilon_x) \subseteq A$$

But because $x_n \to x$

$$\exists\, n_{\varepsilon_x} \in \mathbb{N} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

i.e. almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{O}_d(x, \varepsilon_x)$ and since $\mathcal{O}_d(x, \varepsilon_x) \subseteq A$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A. Since A was chosen arbitrarily, this holds for all such $A \subseteq \tau_d(x)$.

For the converse, suppose that $\forall A \in \tau_d(x)$ almost every element of $(x_n)_{n\in\mathbb{N}}$ belongs to A.

Since $\mathcal{O}_d(x, \varepsilon)$ is a d-open subset of X for all $\varepsilon > 0$ and it includes x, it follows that $\mathcal{O}_d(x, \varepsilon) \in \tau_d(x)$, which drives the result.

Lemma

Let (X, d) be a metric space. Every d-convergent sequence in X has a unique d-limit.

Proof

Let $(x_n)_{n\in\mathbb{N}}$ be a d-convergent sequence in X with $x = d - \lim(x_n)$, $y = d - \lim(x_n)$, and $x \neq y$.

By separateness we know that $d(x, y) > 0$ so there exist $0 < \varepsilon_x < d(x, y)$ and $0 < \varepsilon_y < d(x, y)$ such that

$$\mathcal{O}_d(x, \varepsilon_x) \bigcap \mathcal{O}_d(y, \varepsilon_y) = \varnothing$$

Since $x_n \to x$ for this particular ε_x

$$\exists\, n_{\varepsilon_x} : \forall\, n \geq n_{\varepsilon_x} \quad x_n \in \mathcal{O}_d(x, \varepsilon_x)$$

thus only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(x, \varepsilon_x)$.

Analogously, since $x_n \to y$ only a finite number of elements of the sequence may lie outside of $\mathcal{O}_d(y, \varepsilon_y)$.

But $\mathcal{O}_d(x, \varepsilon_x)$ and $\mathcal{O}_d(y, \varepsilon_y)$ are disjoint sets. Contradiction!

Observe that we need the separateness property to prove the uniqueness of d-limits. Thus, d-limits are not necessarily unique in pseudometric spaces.

Furthermore, the uniqueness of limits cannot necessarily be generalized in topological spaces that are not generated by a metric. Consider the example of the indiscrete topological space where the carrier set is not a singleton, e.g. (X, τ_I) with $X = \{a, b\}$, $a \neq b$, and $\tau_I = \{\varnothing, X\}$. Consider any sequence, $(x_n)_{n\in\mathbb{N}}$, in X. Here, a is a limit of $(x_n)_{n\in\mathbb{N}}$ because the neighbouring system of a with respect to τ_I comprises only of X itself

$$\tau_I(a) = \{X\}$$

and the entire sequence is in X.

Similarly, $\tau_I(b) = \{X\}$ and b is a limit. So in indescrete spaces limits may not be unique.

Definition

Let (X, d_X) and (Y, d_Y) be metric spaces and f a function from X to Y, $f : X \to Y$. f is d_Y/d_X-continuous at $x \in X$ iff

$$\forall (x_n)_{n\in\mathbb{N}} \text{ in } X \text{ with } x = d_X - \lim(x_n) \Rightarrow f(x) = d_Y - \lim(f(x_n))$$

Lemma

Let $f : X \to Y$ be some function with (X, d_X) and (Y, d_Y) metric spaces. Then the following statements are all equivalent to one another (i.e. when any one of them holds, all them concurrently hold) for any point $x \in X$:

1. f is d_Y/d_X-continuous at $x \in X$

2. $\forall \delta > 0, \exists \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$

3. If $A \in \tau_{d_Y}(f(x))$ then $\exists B \in \tau_{d_X}(x) : B \subseteq f^{-1}(A)$

Proof

We want to show that each of the above conditions is a *necessary* and *sufficient* condition for all of the others.

First, we show that 2. is a *necessary* and *sufficient* condition for 1.

Assume that 2. holds. Then $\forall \delta > 0, \exists\, \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$.

For $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $x_n, x \in X$ consider $(f(x_n))_{n\in\mathbb{N}}$. Then for some $\delta > 0$, choose a ε_δ that satisfies 2. Because $x_n \to x$

$$\forall n \geq n^*(\varepsilon_\delta),\, x_n \in \mathcal{O}_{d_X}(x, \varepsilon_\delta) \Rightarrow$$
$$\forall n \geq n^*(\varepsilon_\delta),\, f(x_n) \in f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right)$$

and because we assumed that $f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$

$$\forall n \geq n^*(\varepsilon_\delta),\;\; f(x_n) \in \mathcal{O}_{d_Y}(f(x), \delta)$$

and since δ is arbitrary $f(x_n) \to f(x)$. Because $(x_n)_{n\in\mathbb{N}}$ is arbitrary f is d_Y/d_X-continuous at $x \in X$.

So the 2. is a *sufficient* condition for 1. at x.

Now, suppose that 1. holds (f is d_Y/d_X-continuous at $x \in X$), but for some $\delta > 0$ no ε_δ exists that satisfies 2. and $\forall \varepsilon > 0, f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x), \delta)$. This can be equivalently expressed as

$$\exists \delta > 0 : \forall \varepsilon > 0, f(\mathcal{O}_{d_X}(x, \varepsilon)) \bigcap \mathcal{O}'_{d_Y}(f(x), \delta) \neq \varnothing$$

This implies that

$$\exists \delta > 0 : \forall n \in \mathbb{N}, f(\mathcal{O}_{d_X}(x, \frac{1}{n+1})) \bigcap \mathcal{O}'_{d_Y}(f(x), \delta) \neq \varnothing$$

(since all $n \in \mathbb{N}$ give suitable ε). Consider the images of these sets through f^{-1} (which are also non-empty)

$$f^{-1}\left(f\left(\mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right)\right) \bigcap \mathcal{O}'_{d_Y}\left(f(x), \delta\right)\right) = \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right) \bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x), \delta)) \neq \varnothing, \forall n \in \mathbb{N}$$

and a sequence, $(x_n)_{n\in\mathbb{N}}$, such that the n-th element of the sequence belongs to the n-th such set

$$\begin{aligned} x_n &\in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow \\ x_n &\in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right) \end{aligned}$$

which can be shown to imply a convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x. Thus, $x = d_X - \lim(x_n)$.

By the assumed d_Y/d_X-continuity of f at x, we get $f(x) = d_Y - \lim(f(x_n))$, but

$$\begin{aligned} x_n &\in \mathcal{O}_{d_X}\left(x, \frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \Rightarrow \\ x_n &\in f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)) \iff \\ f(x_n) &\in f(f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))) = \mathcal{O}'_{d_Y}(f(x),\delta) \iff \\ f(x_n) &\notin \mathcal{O}_{d_Y}(f(x),\delta) \end{aligned}$$

which can be shown to make convergence of $(f(x_n))_{n\in\mathbb{N}}$ at $f(x) \in Y$ impossible. Hence we have a contradiction.

So 2. is a *necessary* condition for 1. at x.

Now, we show that 2. and 3. imply one another.

Let 3. hold.

For $\delta > 0$ choose $A = \mathcal{O}_{d_Y}(f(x),\delta)$. By our assumption $\exists$ B in the neighbourhood system $\tau_{d_X}(x)$ such that $B \subseteq f^{-1}(A)$. Since $B \in \tau_{d_X}(x)$ there always exists a $\varepsilon > 0$ such that B is a subset of a d_X-open ball with center x and radius ε. All this implies that

$$\begin{aligned} \mathcal{O}_{d_X}(x,\varepsilon) &\subseteq B \subseteq f^{-1}(A) \Rightarrow \\ \mathcal{O}_{d_X}(x,\varepsilon) &\subseteq f^{-1}(\mathcal{O}_{d_Y}(f(x),\delta)) \Rightarrow \\ f(\mathcal{O}_{d_X}(x,\varepsilon)) &\subseteq f(f^{-1}(\mathcal{O}_{d_Y}(f(x),\delta))) \Rightarrow \\ f(\mathcal{O}_{d_X}(x,\varepsilon)) &\subseteq \mathcal{O}_{d_Y}(f(x),\delta) \end{aligned}$$

So 3. implies 2.

Now, let 2. hold.

Suppose that $\exists A \in \tau_{d_Y}(f(x))$ such that $\forall B \in \tau_{d_X}(x)$, B is not a subset of $f^{-1}(A)$, i.e.

$$B \bigcap \left(f^{-1}(A)\right)' \neq \varnothing \qquad \forall B \in \tau_{d_X}(x)$$

Because all d_X-open balls with center x belong to $\tau_{d_X}(x)$

$$\begin{aligned}
\mathcal{O}_{d_X}(x,\varepsilon) \bigcap \left(f^{-1}(A)\right)' \neq \varnothing && \forall\varepsilon > 0 \Rightarrow \\
\mathcal{O}_{d_X}(x,\varepsilon) \bigcap f^{-1}(A') \neq \varnothing && \forall\varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon) \bigcap f^{-1}(A')\right) \neq \varnothing && \forall\varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap f\left(f^{-1}(A')\right) \neq \varnothing && \forall\varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap A' \neq \varnothing && \forall\varepsilon > 0
\end{aligned}$$

But $A \in \tau_{d_Y}(f(x))$ so there always exists $\delta > 0 : \mathcal{O}_{d_Y}(f(x),\delta) \subseteq A$. This implies that $(\mathcal{O}_{d_Y}(f(x),\delta))' \supseteq A'$ and thus

$$f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap (\mathcal{O}_{d_Y}(f(x),\delta))' \neq \varnothing \qquad \forall\varepsilon > 0$$

which is equivalent to $f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \nsubseteq \mathcal{O}_{d_Y}(f(x),\delta)$ and contradicts 2.

So 2. implies 3.