# MSc Course: Mathematical Economics Optional Exercises-June 2024 Exam

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- The proposed solutions can only be submitted via the relevant assignment at the course's e-class. Details about the submission process can be also found there.
- The proposed solutions will be graded. This grade will be added to the grade of the exam in order for the final grade to be formed. The maximal grade that can be attributed to the proposed solutions is 2 (out of 10).
- The instructor retains the right to ask for clarifications on the proposed solutions before finalizing your grade.
- The grade of the proposed solutions remains valid only for the June 2024 exam period. If a resit exam is needed then a different assignment will be designed specific to the particular exam with the relevant details announced in time.

**Exercise 1.** Given an $X$-valued sequence $(x_n)_{n\in\mathbb{N}}$, define a $(x_n)_{n\in\mathbb{N}}$- subsequence to be any infinite subset of $(x_n)_{n\in\mathbb{N}}$. Prove that if a metric space is totally bounded then every sequence has a Cauchy subsequence. Does the converse also hold?

**Exercise 2.** $(X,d)$ is compact iff it is $d$-totally bounded and $d$-complete. Prove that if a metric space is compact then every sequence has a convergent subsequence. Does the converse also hold?

**Exercise 3.** Consider $(B(\mathbb{N},\mathbb{R}), d_{\sup})$ and $A = \left\{ (x_n)_{n\in\mathbb{N}},\ x_n = \begin{cases} 1, & n = i \\ 2, & n \neq i \end{cases},\ i \in \mathbb{N} \right\} \subset B(\mathbb{N},\mathbb{R})$. Show that $A$ is $d_{\sup}$ - bounded but not $d_{\sup}$ - totally bounded.

**Exercise 4.** Suppose that $(X,d)$ is compact. Prove that if $f : X \to \mathbb{R}$ is $d_u/d$ - continuous then it is bounded and thereby conclude that $C(X,\mathbb{R}) \subseteq B(X,\mathbb{R})$, where $C(X,\mathbb{R}) = \{f : X \to \mathbb{R},\ f \text{ is } d_u/d\text{- continuous}\}$. Prove that $C(X,\mathbb{R})$ is a closed subset of $(B(X,\mathbb{R}), d_{\sup})$. Prove that $C(X,\mathbb{R})$ is $d_{\sup}$- complete.

**Exercise 5.** Suppose that $(X,d)$ is compact. Let $f \in C(X,\mathbb{R})$. Show that $\arg\max_{x\in X} f \neq \emptyset$. Suppose that $x_0$ is the unique maximizer of $f$ over $X$. Show that $x_0$ is distinguishable, i.e. $\forall \varepsilon > 0$, $\sup_{x\in O'_d(x_0,\varepsilon)} f(x) < f(x_0)$.

**Exercise 6.** Suppose that $(X,d)$ is compact. Show that for $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, if the sequence $(f_n)$ is $d_u/d$ - equi - Lipschitz then it is $d_{\sup}$ - bounded. Prove that if furthermore $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$, for some $f : X \to \mathbb{R}$, then also $d_{\sup}(f_n, f) \to 0$ as $n \to \infty$, and conclude that the limit $f$ is $d_u/d$ - Lipschitz.

**Exercise 7.** Given that $(\mathbb{R}, d_u)$ is complete, show that $(\mathbb{R}^n, d_A)$ is complete for any $n > 0$ and $A$ any positive definite $n \times n$ matrix.

**Exercise 8.** Prove the Matkowski Fixed Point Theorem:

**Theorem.** *Suppose that $(X, d)$ is complete, $f : X \to X$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that:*

1. *$g$ is non-decreasing,*
2. *$g$ is continuous at zero,*
3. *$g(t) = 0$ iff $t = 0$,*
4. *$\lim_{m\to\infty} g^{(m)}(t) = 0$, $\forall t \in \mathbb{R}_+$, and*
5. *$\forall t > 0$, $\lim_{m\to\infty} \frac{g^{(m+1)}(t)}{g^{(m)}(t)} = c_t < 1$.*

*Then if $\forall x, y \in X$, $d(f(x), f(y)) \leq g(d(x, y))$, $f$ has a unique fixed point, say $x^\star = \lim_{m\to\infty} f^{(m)}(x)$, for all $x \in X$.*

Show that this is a generalization of BFPT.

**Exercise 9.** (Fredholm Integral Equation of the second kind.) Consider $X = C([a, b], \mathbb{R})$ with $d = d_{\sup}$. Suppose that $\omega : [a, b] \times [a, b] \to \mathbb{R}$ is continuous, that $\omega(x, y) \geq 0$, $\forall x, y \in [a, b]$, that $0 < M_\omega := \sup_{x,y\in[a,b]} \omega(x, y)$, $h \in X$ and let $\lambda > 0$. Consider the integral equation

$$f(x) = h(x) + \lambda \int_a^b \omega(x, y) f(y)\, dy,\ \forall x \in [a, b]. \tag{1}$$

Show that there exists a unique $f \in X$ that satisfies (1) if $\lambda < \frac{1}{M_\omega(b-a)}$.

**Exercise 10.** (Perron-Frobenius) Remember that $A = (a_{i,j})_{i=1,\ldots,q, j=1,\ldots,p}$ with $a_{i,j} \in \mathbb{R}$, $\forall i, j$, is called positive ($A > 0$) iff $a_{i,j} > 0$, $\forall i, j$. Show that if $p = q$ and $A > 0$ then $A$ has at least one positive eigenvalue and at least one positive eigenvector. (Hint: study and use the Brouwer FPT)

**Exercise 11.** Suppose that $X \neq \emptyset$, $(Y, d_Y)$ is complete and consider $(B(X, Y), d^Y_{\sup})$, with $d^Y_{\sup}(f, g) := \sup_{x\in X} d_Y(f(x), g(x))$. Consider $(f_n)_{n\in\mathbb{N}}, f$, with $f_n, f \in B(X, Y), \forall n \in \mathbb{N}$. Show that if (a). $f_n(x) \to f(x)$, $\forall x \in X$, (b). there exists a metric $d_X$ such that $(X, d_X)$ is totally bounded, and (c). $\exists L > 0 : \forall x, y \in X, d_Y(g(x), g(y)) \leq L d_X(x, y)$, $g = f_n, \forall n \in \mathbb{N}$, or $g = f$, then $d^Y_{\sup}(f_n, f) \to 0$.

**Exercise 12.** Suppose that $(B, d_B)$ is compact, $X_1, X_2, \ldots, X_n$ are iid random variables, and for $f : B \times \mathbb{R} \to \mathbb{R}$, (a). $f(\beta, x) \in C(B, \mathbb{R})$, $\forall x \in \mathbb{R}$, and (b). $\mathbb{E}(f(\beta, X_0)) \in C(B, \mathbb{R})$ and has a unique maximiser, say $\beta_0 \in B$. Show that if $\beta_n \in \arg\max_{\beta\in B} \frac{1}{n}\sum_{i=1}^n f(\beta, X_i)$, then $d_B(\beta_n, \beta_0) \to 0$ in probability. (*Hint: You can among others use the definitions and results of exercises 2, 4 and 5, as well as the ULLN and the approximation of optimization problems results -wherever necessary appropriately modified- proven in the classroom.*)