MSc Course: Mathematical Analysis - Optional Exercises (2021-22)

Stelios Arvanitis

- The following set of exercises is optional.
- The details of the (electronic) submission process will be announced in time.
- The instructor retains the right to ask for clarifications on the proposed solutions.

Exercise 1. Given an X-valued sequence $(x_n)_{n\in\mathbb{N}}$, define a $(x_n)_{n\in\mathbb{N}}$- subsequence to be any infinite subset of $(x_n)_{n\in\mathbb{N}}$. Prove that if a metric space is totally bounded then every sequence has a Cauchy subsequence. Does the converse also hold?

Exercise 2. (X,d) is compact iff it is d-totally bounded and d-complete. Prove that if a metric space is compact then every sequence has a convergent subsequence. Does the converse also hold?

Exercise 3. Consider $(B(\mathbb{N},\mathbb{R}), d_{\sup})$ and $A = \left\{ (x_n)_{n\in\mathbb{N}},\ x_n = \begin{cases} 1, & n = i \\ 2, & n \neq i \end{cases},\ i \in \mathbb{N} \right\} \subset B(\mathbb{N},\mathbb{R})$. Show that A is $d_{\sup}$ - bounded but not $d_{\sup}$ - totally bounded.

Exercise 4. Suppose that (X,d) is compact. Prove that if $f : X \to \mathbb{R}$ is d_I/d - continuous then it is bounded and thereby conclude that $C(X,\mathbb{R}) \subseteq B(X,\mathbb{R})$, where $C(X,\mathbb{R}) = \{f : X \to \mathbb{R},\ f \text{ is } d_I/d\text{- continuous}\}$. Prove that $C(X,\mathbb{R})$ is a closed subset of $(B(X,\mathbb{R}), d_{\sup})$. Prove that $C(X,\mathbb{R})$ is $d_{\sup}$- complete.

Exercise 5. Suppose that (X,d) is compact. Let $f \in C(X,\mathbb{R})$. Show that $\arg\max_{x\in X} f \neq \emptyset$. Suppose that x_0 is the unique maximizer of f over X. Show that x_0 is distinguishable, i.e. $\forall\varepsilon > 0$, $\sup_{x\in O_d'(x_0,\varepsilon)} f(x) < \sup_{x\in X} f(x)$.

Exercise 6. Suppose that (X,d) is compact. Show that for $f_n : X \to \mathbb{R}$, $n \in \mathbb{N}$, if the sequence (f_n) is d_u/d - equi - Lipschitz then it is $d_{\sup}$ - bounded. Prove that if furthermore $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$, for some $f : X \to \mathbb{R}$, then also $d_{\sup}(f_n, f) \to 0$ as $n \to \infty$, and conclude that the limit f is d_u/d - Lipschitz. (d_u denotes the usual metric)

Exercise 7. Given that $(\mathbb{R}, d_I)$ is complete, show that $(\mathbb{R}^n, d_A)$ is complete for any $n > 0$ and A any positive definite $n \times n$ matrix.

Exercise 8. Prove the Matkowski Fixed Point Theorem:

Theorem. *Suppose that* (X,d) *is complete,* $f:X\to X$*, and* $g:\mathbb{R}_+\to\mathbb{R}_+$ *such that:*

1. g *is non-decreasing,*
2. g *is continuous at zero,*
3. $g(t)=0$ *iff* $t=0$*,*
4. $\lim_{m\to\infty} g^{(m)}(t)=0$*,* $\forall t\in\mathbb{R}_+$*, and*
5. $\forall t>0$*,* $\lim_{m\to\infty}\frac{g^{(m+1)}(t)}{g^{(m)}(t)}=c_t<1$*.*

Then if $\forall x,y\in X$*,* $d(f(x),f(y))\leq g(d(x,y))$*,* f *has a unique fixed point, say* $x^\star=\lim_{m\to\infty} f^{(m)}(x)$*, for all* $x\in X$*.*

Show that this is a generalization of BFPT.

Exercise 9. (Fredholm Integral Equation of the second kind.) Consider $X=C([a,b],\mathbb{R})$ with $d=d_{\text{sup}}$. Suppose that $\omega:[a,b]\times[a,b]\to\mathbb{R}$ is continuous, that $\omega(x,y)\geq 0$, $\forall x,y\in[a,b]$, that $0<M_\omega\coloneqq\sup_{x,y\in[a,b]}\omega(x,y)$, $h\in X$ and let $\lambda>0$. Consider the integral equation

$$f(x)=h(x)+\lambda\int_a^b \omega(x,y)\,f(y)\,dy,\ \forall x\in[a,b]\,. \tag{1}$$

Show that there exists a unique $f\in X$ that satisfies (1) if $\lambda<\frac{1}{M_\omega(b-a)}$.

Exercise 10. (Perron-Frobenius) Remember that $A=(a_{i,j})_{i=1,\ldots,q,j=1,\ldots,p}$ with $a_{i,j}\in\mathbb{R}$, $\forall i,j$, is called positive ($A>0$) iff $a_{i,j}>0$, $\forall i,j$. Show that if $p=q$ and $A>0$ then A has at least one positive eigenvalue and at least one positive eigenvector. (Hint: study and use the Brouwer FPT)