Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
Lemma on Bounded Functional Spaces and Completeness
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Lemma

If (X, d) is a complete metric space and $Y \neq \varnothing$ is a non-empty set, then the structured set $(\mathcal{B}(Y, X), d^d_{sup})$ is a complete metric space, with $d^d_{sup}(f, g) = \sup_{x \in Y} d(f(x), g(x))$, $\forall f, g \in \mathcal{B}(Y, X)$.

Proof

We need to show that:

α) $(\mathcal{B}(Y, X), d^d_{sup})$ is a metric space, which requires that:

- $\mathcal{B}(Y, X)$ be a non-empty set.
- d^d_{sup} be a metric function on $\mathcal{B}(Y, X)$ (properties i-iv).

β) Every d^d_{sup}-Cauchy sequence on $\mathcal{B}(Y, X)$ has a d^d_{sup}-limit in $\mathcal{B}(Y, X)$, which requires that for all $(f_n)_{n \in \mathbb{N}} : f_n \in \mathcal{B}(Y, X)$, $\forall n \in \mathbb{N}$, there exists a function, f, such that:

- $f = d^d_{sup} - \lim f_n$
- $f \in \mathcal{B}(Y, X)$

α) Since $Y \neq \varnothing$ we can define at least one function (if not more) that maps elements of Y to elements of X (also non-empty). For example, the constant function $f_c : Y \to X$ such that $\forall y \in Y, f_c(y) = x_c$ for some $x_c \in X$.

Also, d^d_{sup} is a metric function on $\mathcal{B}(Y, X)$ since:

i) $\forall f, g \in \mathcal{B}(Y, X)$, $d^d_{sup}(f, g) = \sup_{x \in Y} d(f(x), g(x)) \overset{i}{\geq} 0$

ii) $\forall f, g \in \mathcal{B}(Y, X)$,

$$\begin{aligned}
d^d_{sup}(f,g) = 0 && \Longleftrightarrow \\
\sup_{x\in Y} d(f(x), g(x)) = 0 && \overset{i}{\Longleftrightarrow} \\
d(f(x), g(x)) = 0, \ \forall x \in Y && \overset{ii}{\Longleftrightarrow} \\
f(x) = g(x), \ \forall x \in Y && \Longleftrightarrow \\
f = g &&
\end{aligned}$$

iii) $\forall f, g \in \mathcal{B}(Y, X)$, $d^d_{sup}(f,g) = \sup_{x\in Y} d(f(x), g(x)) \overset{iii}{=} \sup_{x\in Y} d(g(x), f(x)) = d^d_{sup}(g, f)$

iv) $\forall f, g, h \in \mathcal{B}(Y, X)$,

$$\begin{aligned}
d^d_{sup}(f,g) &= \sup_{x\in Y} d(f(x), g(x)) \\
&\leq \sup_{x\in Y} \ (d(f(x), h(x)) + d(h(x), g(x))) \\
&\leq \sup_{x\in Y} d(f(x), h(x)) + \sup_{x\in Y} d(h(x), g(x)) \\
&= d^d_{sup}(f,h) + d^d_{sup}(h,g)
\end{aligned}$$

So d^d_{sup} is a metric function on $\mathcal{B}(Y, X)$ and $(\mathcal{B}(Y, X), d^d_{sup})$ is a metric space.

β) Consider an arbitrary d^d_{sup}-Cauchy sequence on $\mathcal{B}(Y, X)$, $(f_n)_{n\in\mathbb{N}} : f_n \in \mathcal{B}(Y, X) \, \forall n \in \mathbb{N}$. Then $\forall \varepsilon > 0 \, \exists \, n(\varepsilon)$ such that

$$\begin{aligned}
d^d_{sup}(f_n, f_m) < \varepsilon, \ \forall n, m > n(\varepsilon) \\
\sup_{x\in Y} d(f_n(x), f_m(x)) < \varepsilon, \ \forall n, m > n(\varepsilon) \\
\forall x \in Y, \ d(f_n(x), f_m(x)) < \varepsilon, \ \forall n, m > n(\varepsilon)
\end{aligned}$$

so that $(f_n(x))_{n\in\mathbb{N}}$ is a d-Cauchy sequence on X, $\forall x \in Y$. Furthermore, because (X, d) is complete, $(f_n(x))_{n\in\mathbb{N}}$ converges to a d-limit in X, say $\phi_x \in X$, for all $x \in Y$.

So starting from a d^d_{sup}-Cauchy sequence on $\mathcal{B}(Y, X)$ we can generate a collection of d-convergent sequences on X (one sequence for each $x \in Y$). That is, we have that

$$\begin{aligned}
&\forall \ (f_n)_{n\in\mathbb{N}} : (f_n)_{n\in\mathbb{N}} \ d^d_{sup}\text{-Cauchy on } \mathcal{B}(Y, X) \\
&\exists \ \{(y_n)_{n\in\mathbb{N}} : y_n = f_n(x) \, \forall n \in \mathbb{N}, (y_n)_{n\in\mathbb{N}} \ d\text{-convergent on } X, \ \forall x \in Y\}
\end{aligned}$$

We can define a function, $f : Y \to X$, for any such starting sequence and subsequent collection such that $f(x) \coloneqq \phi_x$, $\forall x \in Y$. To show that $(\mathcal{B}(Y, X), d^d_{sup})$ is complete, it suffices to show that $d^d_{sup} - \lim f_n = f$ and that $f \in \mathcal{B}(Y, X)$.

Firstly, consider $d(f_n(x), f_m(x))$ and think of it as a sequence, $(z_m)_{m\in\mathbb{N}}$, for $n \in \mathbb{N}$ given. We have that

$$\begin{aligned}
d^d_{sup}(f_n, f) &= \sup_{x\in Y} d(f_n(x), f(x)) \\
&= \sup_{x\in Y} d(f_n(x), d - \lim_{m\to+\infty} f_m(x)) \\
&\underset{\text{continuous}}{\overset{d \text{ is}}{=}} \sup_{x\in Y} d - \lim_{m\to+\infty} d(f_n(x), f_m(x)) \\
&\leq \sup_{x\in Y} \sup_{m\geq n} d(f_n(x), f_m(x)) \\
&\leq \sup_{x\in Y} \sup_{m\geq n} \sup_{x\in Y} d(f_n(x), f_m(x)) \\
&= \sup_{m\geq n} \sup_{x\in Y} d(f_n(x), f_m(x)) \\
&= \sup_{m\geq n} d^d_{sup}(f_n, f_m)
\end{aligned}$$

and also

$$\begin{aligned}
&(f_n)_{n\in\mathbb{N}} \text{ is } d^d_{sup}\text{-Cauchy} \\
&\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : d^d_{sup}(f_n, f_m) < \varepsilon && \forall n, m \geq n(\varepsilon) \\
&\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : \sup_{m\geq n} d^d_{sup}(f_n, f_m) < \varepsilon && \forall n \geq n(\varepsilon) \\
&\forall\varepsilon > 0,\ \exists\, n(\varepsilon) : d^d_{sup}(f_n, f) < \varepsilon && \forall n \geq n(\varepsilon)
\end{aligned}$$

thus as $n \to +\infty$, $f_n \to f$ with respect to d^d_{sup}.

Secondly, for any two points $f(x), f(y) \in X$ and some $n \in \mathbb{N}$ we have that

$$\begin{aligned}
\sup_{x,y\in Y} d(f(x), f(y)) &\leq \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f(y)) \\
&\leq \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&\leq \sup_{x\in Y} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{y\in Y} d(f_{n(\varepsilon)}(y), f(y)) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&= 2\sup_{x\in Y} d(f_{n(\varepsilon)}(x), f(x)) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y)) \\
&= 2d^d_{sup}(f_{n(\varepsilon)}, f) + \sup_{x,y\in Y} d(f_{n(\varepsilon)}(x), f_{n(\varepsilon)}(y))
\end{aligned}$$

where $n(\varepsilon)$ is such that $d^d_{sup}(f_{n(\varepsilon)}, f) < \varepsilon$. This $n(\varepsilon)$ exists since $f_n \to f$ with respect to d^d_{sup}. Thus the first additive term is bounded by 2ε. The second additive term is the maximum distance between all values of $f_{n(\varepsilon)}$ on X. Since $f_{n(\varepsilon)} \in \mathcal{B}(Y, X)$ this number is also bounded. Thus, $\sup_{x,y\in Y} d(f(x), f(y)) < +\infty$, which establishes that $f \in \mathcal{B}(Y, X)$, i.e. f is a bounded X-valued function.

So for (X, d) complete space, every d^d_{sup}-Cauchy sequence on $\mathcal{B}(Y, X)$ is d^d_{sup}-convergent in $\mathcal{B}(Y, X)$.

Thus, if (X, d) is a complete metric space, then $(\mathcal{B}(Y, X), d^d_{sup})$.