Athens University of Economics and Business
Department of Economics

Postgraduate Program - MSc in Economic Theory
Course: Mathematical Economics (Mathematics II)
Some Solutions to Exercises 3
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Exercise 2

Suppose that $X = \{f : [a,b] \to \mathbb{R} : \int_a^b f^2(x)dx < +\infty\}$, with $a < b$ real numbers. Also consider the metric function $d(f,g) := \left(\int_a^b (f(x) - g(x))^2 dx\right)^{\frac{1}{2}}$ on X. For $\mathbf{0} : [a,b] \to \mathbb{R}, \mathbf{0}(x) := 0\ \forall x \in [a,b]$, consider $\mathcal{O}_d[\mathbf{0}, 1]$ and show that it is not d-totally bounded.

Solution

The d-closed ball centered at $\mathbf{0} \in X$ with radius 1 is a set such that

$$\begin{aligned}
\mathcal{O}_d[\mathbf{0}, 1] &= \{g \in X : d(\mathbf{0}, g) \leq 1\} \\
&= \{g \in X : \left(\int_a^b (\mathbf{0}(x) - g(x))^2 \, dx\right)^{\frac{1}{2}} \leq 1\} \\
&= \{g \in X : \left(\int_a^b g^2(x)dx\right)^{\frac{1}{2}} \leq 1\}
\end{aligned}$$

$\forall \varepsilon \geq 1,\ \mathcal{O}_d[\mathbf{0}, 1] \subseteq \mathcal{O}_d[\mathbf{0}, \varepsilon]$ and so the whole set can be covered by a collection of one closed ball ($\mathcal{O}_d[\mathbf{0}, \varepsilon]$).

For $\varepsilon < 1$ consider the following sequence of functions in $\mathcal{O}_d[\mathbf{0}, 1]$

$$(g_n)_{n \in \mathbb{N}} : g_n \in \mathcal{O}_d[\mathbf{0}, 1]\ \forall n \in \mathbb{N}, d(g_m, g_n) > \frac{1}{2}\ \forall m \neq n \in \mathbb{N}$$

Using a modified Riesz's Lemma* for functional spaces, it can be shown that such a sequence exists.

Observe that $G = \{f : [a,b] \to \mathbb{R} : f = g_n \forall n \in \mathbb{N}\}$ is a subset of $\mathcal{O}_d[\mathbf{0}, 1]$.

Suppose that G is a d-totally bounded subset of X. Then for $\varepsilon = \dfrac{1}{4}$ there should exist a <u>finite</u> number of d-closed balls in X with radius $\dfrac{1}{4}$ that collectively include all of the <u>infinitely many</u> functions in G and thus cover G. By the Pigeonhole Principle** each such ball must include more than one element of G. But then, for some $m \neq n$ this would mean that $d(g_m, g_n) \leq \dfrac{1}{4} < \dfrac{1}{2}$. Contradiction!

So G cannot be a d-totally bounded subset of X and since $G \subseteq \mathcal{O}_d[\mathbf{0}, 1]$ we have that $\mathcal{O}_d[\mathbf{0}, 1]$ is not a d-totally bounded subset of X.

(*) Riesz's Lemma

For (X, d) normed vector space (i.e. the metric d is a p-norm), $(S, d|_{S\times S})$ non-dense linear subspace of (X, d), and $0 < \varepsilon < 1$, there exists $x \in X$ *of unit norm* (i.e. $d(\mathbf{0}, x) = ||x||_p = 1$) such that $d(x, s) \geq 1 - \varepsilon, \forall s \in S$.

(**) Pigeonhole Principle

For $n, m, k \in \mathbb{N}$ with $n = km + 1$, if we distribute n elements across m sets then at least one set will contain at least $k + 1$ elements.

<u>Remarks:</u>

- $\exists$ an infinite number of $f \in X \setminus \mathcal{O}_d[\mathbf{0}, 1]$. E.g. consider $f_n(x) = n\sqrt{\frac{1}{b-a}}, \forall x \in [a, b], \forall n \in \mathbb{N}$.
- $\exists$ an infinite number of $f \in \mathcal{O}_d[\mathbf{0}, 1]$. E.g. consider $f_n(x) = \frac{1}{n}\sqrt{\frac{1}{b-a}}, \forall x \in [a, b], \forall n \in \mathbb{N}$.

Exercise 4

For (X, d) a general metric space, $x, y \in X$, and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} : x_n, y_n \in X \ \forall n \in \mathbb{N}$, with $x = d-\lim(x_n)$ and $y = d-\lim(y_n)$. Given that $d(x_n, y_y) \to d(x, y)$ w.r.t the usual metric on $\mathbb{R}$, d_u, show that, given y, $f = d(\cdot, y) : X \to \mathbb{R}$ is d_u/d-continuous. Conclude analogously for $g = d(y, \cdot)$.

Solution

$f(x; y) = d(x, y), y \in X, \forall x \in X$ is d_u/d-continuous at $x \in X$ because, by the fact that $d_u - \lim d(x_n, y_n) = d(x, y)$, <u>for all</u> sequences $(x_n)_{n\in\mathbb{N}}$ in X with $d - \lim x_n = x$ the corresponding sequence $(f(x_n; y))_{n\in\mathbb{N}}$ in $\mathbb{R}$ has $d_u - \lim f(x_n; y) = d_u - \lim d(x_n, y_n) = d(x, y) = f(x; y)$. Since $x \in X$ is arbitrary, f is d_u/d-continuous in X.

By symmetry of d, $g = d(\cdot, x)$ is d_u/d-continuous, since $f = d(x, \cdot)$ is d_u/d-continuous.

Lemma: Total Boundness and Finite Products

Let (X_i, d_i) metric spaces $\forall i \in \mathcal{I}$ with $\mathcal{I}$ a finite index set. For the cartesian product $X \coloneqq \prod_{i\in\mathcal{I}} X_i$ there can be defined the following structured sets (X, d_Π), $d_\Pi \in \{d_{\Pi_{max}}, d_{\Pi_I}, d_{\Pi_{||}}\}$ and d_Π are defined as

$$d_{\Pi_{max}} = \max_{i\in\mathcal{I}} d_i$$
$$d_{\Pi_I} = \left(\sum_{i\in\mathcal{I}} d_i^2\right)^{\frac{1}{2}}$$
$$d_{\Pi_{||}} = \sum_{i\in\mathcal{I}} d_i$$

and are appropriate metric functions on X. Let $A_i \subseteq X_i, \forall i \in \mathcal{I}$ and $A \coloneqq \prod_{i\in\mathcal{I}} A_i$, which implies that $A \subseteq X$. *Then A is a $d_{\Pi_{max}}(/d_{\Pi_I}/d_{\Pi_{||}})$-totally bounded subset of X iff A_i are d_i-totally bounded subsets of X_i $\forall i \in \mathcal{I}$.*

Proof

It suffices to show that

$$A \text{ is } d_{\Pi_{max}}\text{-totally bounded subset of } X \iff A_i \text{ is } d_i\text{-totally bounded subset of } X_i\ \forall i \in \mathcal{I}$$

because

$$d_{\Pi_{max}} \leq d_{\Pi_I} \leq d_{\Pi_{||}} \leq n d_{\Pi_{max}}$$

where $n \in \mathbb{N}^*$ is the number of elements in $\mathcal{I}$, and the total boundeness property for a specific set (say A) is inherited by d_{Π_I} and $d_{\Pi_{||}}$ from $d_{\Pi_{max}}$, and by $d_{\Pi_{max}}$ from the other two.

A compact illustration of the proof is the following (statements above the $\iff$ sign describe how to move forward, while statements below show the way back)

$$
\begin{aligned}
&\forall i \in \mathcal{I},\ A_i \text{ is } d_i\text{-totally bounded subset of } X_i \iff \\
&\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists C_{A_i,\varepsilon_i} := \{\mathcal{O}_{d_i}(x_{ij},\varepsilon_i), x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite}\} : A_i \subseteq \bigcup_{j\in\mathcal{I}_i} \mathcal{O}_{d_i}(x_{ij},\varepsilon_i) \iff \\
&\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, y_i \in \mathcal{O}_{d_i}(x_{ij},\varepsilon_i) \iff \\
&\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, d_i(x_{ij},y_i) < \varepsilon_i \underset{\text{Choose } \varepsilon_i=\varepsilon}{\iff} \\
&\forall i \in \mathcal{I}, \forall \varepsilon_i > 0\ \exists x_{ij} \in X_i, j \in \mathcal{I}_i \text{ finite} : \forall y_i \in A_i, \max_{i\in\mathcal{I}} d_i(x_{ij},y_i) < \max_{i\in\mathcal{I}} \varepsilon_i \overset{\varepsilon:=\max_{i\in\mathcal{I}}\varepsilon_i, \overline{\mathcal{I}}=\bigcup_{i\in\mathcal{I}}\mathcal{I}_i}{\underset{\text{Choose } \mathcal{I}_i=\overline{\mathcal{I}}}{\iff}} \\
&\forall \varepsilon > 0\ \exists x_j \in X, j \in \overline{\mathcal{I}} \text{ finite} : \forall y \in A, d_{\Pi_{max}}(x_j,y) < \varepsilon \iff \\
&\forall \varepsilon > 0\ \exists x_j \in X, j \in \overline{\mathcal{I}} \text{ finite} : \forall y \in A, y \in \mathcal{O}_{d_{\Pi_{max}}}(x_j,\varepsilon) \iff \\
&\forall \varepsilon > 0\ \exists C_{A,\varepsilon} := \{\mathcal{O}_{d_{\Pi_{max}}}(x_j,\varepsilon), x_j \in X, j \in \overline{\mathcal{I}} \text{ finite}\} : A \subseteq \bigcup_{j\in\overline{\mathcal{I}}} \mathcal{O}_{d_{\Pi_{max}}}(x_j,\varepsilon) \iff \\
&A \text{ is } d_{\Pi_{max}}\text{-totally bounded subset of } X
\end{aligned}
$$

More verbosely, if A_i are d_i-totally bounded subsets of X_i for all $i \in \mathcal{I}$, then there exist $\forall i$ finite d_i-open covers for any $\varepsilon_i > 0$ (we choose a different ε_i for each A_i).
That means that each element, y_i, of each A_i belongs to some d_i-open ball with radius ε_i and the number of these balls (as well as their centres, x_{ij}) is finite for all i.
We construct elements of X and A using the above x_{ij} and y_i. If we consider the $d_{\Pi_{max}}$ metric on X, we can see that the distance of each $y = (y_n)_{n\in\mathcal{I}}$ in A from each $x_j = (x_{nj})_{n\in\mathcal{I}}$ in X, given by $d_{\Pi_{max}}$, is equal to the greatest distance between their elements, given by the corresponding d_i, i.e. $\forall x_j, y$

$$d_{\Pi_{max}}(x_j,y) = \max_{i\in\mathcal{I}} d_i(x_{ij},y_i)$$

So we can construct $d_{\Pi_{max}}$-open balls using the x_j-s as centres and setting $\varepsilon := \max_{i\in\mathcal{I}}$ as their radii and cover all of A with them. Their number is finite.
Thus, we have constructed a finite $d_{\Pi_{max}}$-open cover of A **for all** $\varepsilon > 0$ using the fact that A_i are d_i-totally bounded subsets of X_i for all $i \in \mathcal{I}$. So A is a $d_{\Pi_{max}}$-totally bounded subset of X.

Conversely, if A is a $d_{\Pi_{max}}$- totally bounded subset of X, then for all $\varepsilon > 0$ there exists a finite cover of $d_{\Pi_{max}}$-open balls with radius ε, such that $\forall y \in A$, y belongs to one of these (finitely many) balls. By definition of $d_{\Pi_{max}}$, the above means that each element of y, y_i, will belong to a d_i-open ball of radius ε. For all i the union of these balls covers each A_i and their number is the same as the number of balls used to cover A, which is finite.

Thus we have constructed finite d_i-open covers of A_i and A_i are d_i-totally bounded subsets of X_i for all i.

A few remarks:

- For a subset in a metric space to be totally bounded, a finite cover must exist **for all radii**. Make sure you see that this is the case here.

- Pay attention to the possibility that the index sets of each cover for the various i may not include the same indices. Thus, when constructing the index set for the cartesian product, we use their union ($\overline{\mathcal{I}} = \bigcup_{i \in \mathcal{I}} \mathcal{I}_i$). This means that we may need to use some x_{ij} that are not necessary to cover A_i, but are needed to construct the x_j that define the balls that cover A.

  E.g. $\mathcal{I} = \{1, 2\}$ and for some $\varepsilon_1, \varepsilon_2 > 0$ the index sets of the finite covers of A_1 and A_2 are $\mathcal{I}_1 = \{1, 2\}$ and $\mathcal{I}_2 = \{1, 2, 3\}$. Then we need $\overline{\mathcal{I}} = \{1, 2, 3\}$ to construct the cover of $A = A_1 \times A_2$ of radius $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\}$.

- The minimum effective size of a finite cover's index set depends on the size of the radius (and will typically converge to infinity as a radius approaches zero). But for all strictly positive radii, these index sets are finite.

Lemma: Continuity via Open Sets (Open Balls and Neighbourhoods)

A function, $f : X \to Y$, with (X, d_X) and (Y, d_Y) metric spaces, is d_Y/d_X-continuous at a point $x \in X$ iff the following equivalent conditions hold:

- $\forall \delta > 0, \exists \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$
- If $A \in \tau_{d_Y}(f(x))$ then $\exists B \in \tau_{d_X}(x) : B \subseteq f^{-1}(A)$

Proof

First, we show that the first point is a *necessary* and *sufficient* condition for d_Y/d_X-continuity of f at x.

Assume that $\forall \delta > 0, \exists \varepsilon_\delta : f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$.
For $(x_n)_{n\in\mathbb{N}}$ such that $x_n \to x$ and $x_n, x \in X$ consider $(f(x_n))_{n\in\mathbb{N}}$. Then for some $\delta > 0$, choose a ε_δ that satisfies our assumption. Because $x_n \to x$

$$\forall n \geq n^*(\varepsilon_\delta),\, x_n \in \mathcal{O}_{d_X}(x, \varepsilon_\delta) \Rightarrow$$
$$\forall n \geq n^*(\varepsilon_\delta),\, f(x_n) \in f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right)$$

and because we assumed that $f\left(\mathcal{O}_{d_X}(x, \varepsilon_\delta)\right) \subseteq \mathcal{O}_{d_Y}(f(x), \delta)$

$$\forall n \geq n^*(\varepsilon_\delta),\ f(x_n) \in \mathcal{O}_{d_Y}(f(x), \delta)$$

and since δ is arbitrary $f(x_n) \to f(x)$. Because $(x_n)_{n\in\mathbb{N}}$ is arbitrary f is d_Y/d_X-continuous at $x \in X$.
So the first point is a *sufficient* condition for d_Y/d_X-continuity of f at x.

Now, suppose that f is d_Y/d_X-continuous at $x \in X$, but for some $\delta > 0$ no ε_δ exists that satisfies the property we want to prove and $\forall \varepsilon > 0, f\left(\mathcal{O}_{d_X}(x, \varepsilon)\right) \not\subseteq \mathcal{O}_{d_Y}(f(x), \delta)$. This can be equivalently expressed as

$$\exists \delta > 0 : \forall \varepsilon > 0, f(\mathcal{O}_{d_X}(x, \varepsilon)) \bigcap \mathcal{O}'_{d_Y}(f(x), \delta) \neq \varnothing$$

This implies that

$$\exists \delta > 0 : \forall n \in \mathbb{N}, f(\mathcal{O}_{d_X}(x, \frac{1}{n+1})) \bigcap \mathcal{O}'_{d_Y}(f(x), \delta) \neq \varnothing$$

(since all $n \in \mathbb{N}$ give suitable ε). Consider the images of these sets through f^{-1} (which are also non-empty)

$$f^{-1}\left(f\left(\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\right)\bigcap \mathcal{O}'_{d_Y}(f(x),\delta)\right)=\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))\neq\varnothing,\forall n\in\mathbb{N}$$

and a sequence, $(x_n)_{n\in\mathbb{N}}$, such that the n-th element of the sequence belongs to the n-th such set

$$\begin{aligned}
&x_n\in\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))\Rightarrow\\
&x_n\in\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)
\end{aligned}$$

which can be shown to imply a convergence of the sequence $(x_n)_{n\in\mathbb{N}}$ to x. Thus, $x=d_X-\lim(x_n)$. By the assumed d_Y/d_X-continuity of f at x, we get $f(x)=d_Y-\lim(f(x_n))$, but

$$\begin{aligned}
&x_n\in\mathcal{O}_{d_X}\left(x,\frac{1}{n+1}\right)\bigcap f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))\Rightarrow\\
&x_n\in f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta))\iff\\
&f(x_n)\in f(f^{-1}(\mathcal{O}'_{d_Y}(f(x),\delta)))=\mathcal{O}'_{d_Y}(f(x),\delta)\iff\\
&f(x_n)\notin\mathcal{O}_{d_Y}(f(x),\delta)
\end{aligned}$$

which can be shown to make convergence of $(f(x_n))_{n\in\mathbb{N}}$ at $f(x)\in Y$ impossible. Hence we have a contradiction.
So the first point is a *necessary* condition for d_Y/d_X-continuity of f at x.

Now, we show that both points imply one another.

Let the second point hold.
For $\delta>0$ choose $A=\mathcal{O}_{d_Y}(f(x),\delta)$. By our assumption $\exists$ B in the neighbourhood system $\tau_{d_X}(x)$ such that $B\subseteq f^{-1}(A)$. Since $B\in\tau_{d_X}(x)$ there always exists a $\varepsilon>0$ such that B is a subset of a d_X-open ball with center x and radius ε. All this implies that

$$\begin{aligned}
\mathcal{O}_{d_X}(x,\varepsilon)&\subseteq B\subseteq f^{-1}(A)\Rightarrow\\
\mathcal{O}_{d_X}(x,\varepsilon)&\subseteq f^{-1}(\mathcal{O}_{d_Y}(f(x),\delta))\Rightarrow\\
f(\mathcal{O}_{d_X}(x,\varepsilon))&\subseteq f(f^{-1}(\mathcal{O}_{d_Y}(f(x),\delta)))\Rightarrow\\
f(\mathcal{O}_{d_X}(x,\varepsilon))&\subseteq\mathcal{O}_{d_Y}(f(x),\delta)
\end{aligned}$$

So the second point implies the first one.

Now, let the first point hold.

Suppose that $\exists A \in \tau_{d_Y}(f(x))$ such that $\forall B \in \tau_{d_X}(x)$, B is not a subset of $f^{-1}(A)$, i.e.

$$B \bigcap \left(f^{-1}(A)\right)' \neq \varnothing \qquad \forall B \in \tau_{d_X}(x)$$

Because all d_X-open balls with center x belong to $\tau_{d_X}(x)$

$$\begin{aligned}
\mathcal{O}_{d_X}(x,\varepsilon) \bigcap \left(f^{-1}(A)\right)' \neq \varnothing & \qquad \forall \varepsilon > 0 \Rightarrow \\
\mathcal{O}_{d_X}(x,\varepsilon) \bigcap f^{-1}(A') \neq \varnothing & \qquad \forall \varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon) \bigcap f^{-1}(A')\right) \neq \varnothing & \qquad \forall \varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap f\left(f^{-1}(A')\right) \neq \varnothing & \qquad \forall \varepsilon > 0 \Rightarrow \\
f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap A' \neq \varnothing & \qquad \forall \varepsilon > 0
\end{aligned}$$

But $A \in \tau_{d_Y}(f(x))$ so there always exists $\delta > 0 : \mathcal{O}_{d_Y}(f(x),\delta) \subseteq A$. This implies that $(\mathcal{O}_{d_Y}(f(x),\delta))' \supseteq A'$ and thus

$$f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \bigcap (\mathcal{O}_{d_Y}(f(x),\delta))' \neq \varnothing \qquad \forall \varepsilon > 0$$

which is equivalent to $f\left(\mathcal{O}_{d_X}(x,\varepsilon)\right) \not\subseteq \mathcal{O}_{d_Y}(f(x),\delta)$ and contradicts our assumption.

So the first point implies the second.