Athens University of Economics and Business
*Department of Economics*

Postgraduate Program - MSc in Economic Theory
*Course: Mathematical Economics (Mathematics II)*
Some Solutions to Exercises 1
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**Exercise 1**

$(X, d_x)$ being a metric space means that $d : X \times X \to \mathbb{R}$ is a suitable metric function on the set $X$. So $d_x$ satisfies the following properties:

i) $d_x(x, y) \geq 0, \forall x, y \in X$ **(positivity)**

ii) $d_x(x, y) = 0 \iff x = y, \forall x, y \in X$ **(separateness)**

iii) $d_x(x, y) = d_x(y, x), \forall x, y \in X$ **(symmetry)**

iv) $d_x(x, y) \leq d_x(x, z) + d_x(z, y), \forall x, y, z \in X$ (**subadditivity/triangle inequality**)

$f$ being injective means that $f(x) = f(y) \iff x = y, \forall x, y \in Z$.

The necessary and sufficient condition to prove that $(Z, d_f)$ is a metric space is that $d_f$ be a suitable metric on $Z$, i.e. that $d_f$ satisfies the required properties i-iv on $Z$. So we have:

i) $d_f(x, y) = d_x(f(x), f(y)) \overset{i}{\geq} 0, \forall f(x), f(y) \in X.$

Because $f$ is a 1-1 correspondence from $Z$ to $X$, we get that $d_f(x, y) \geq 0, \forall x, y \in Z$.

ii) $d_f(x, y) = 0 \iff d_x(f(x), f(y)) = 0 \overset{ii}{\iff} f(x) = f(y) \overset{f\ 1-1}{\iff} x = y,\ \forall x, y \in Z$

iii) $d_f(x, y) = d_x(f(x), f(y)) \overset{iii}{=} d_x(f(y), f(x)) \overset{f\ 1-1}{=} d_f(y, x),\ \forall x, y \in Z$

iv) $d_f(x, y) = d_x(f(x), f(y)) \overset{iv}{\leq} d_x(f(x), f(z)) + d_x(f(z), f(y)) \overset{f\ 1-1}{=} d_f(x, z) + d_f(z, y),\ \forall x, y, z \in Z$

We have shown that $d_f$ satisfies properties i-iv on $Z$. So $d_f$ is a metric on $Z$ and $(Z, d_f)$ is a metric space.

**Exercise 2**

It holds for all $x, y, z \in X$ that

$$\begin{aligned} |d(x,z) - d(z,y)| &\overset{iv}{\leq} |d(x,y) + d(y,z) - d(z,y)| \\ &\overset{iii}{=} |d(x,y)| \\ &\overset{i}{=} d(x,y) \end{aligned}$$

**Exercise 4**

To show that $e$ is a metric on $X$, we must show that it has the properties i-iv on $X$:

i) $d(x,y) \geq 0 \Rightarrow 1 + d(x,y) > 0, \forall x, y \in X$ so $e$ is a well defined real function and $e(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \geq 0, \forall x, y \in X$ as the ratio of a non-negative number over a positive number.

ii) $e(x,y) = 0 \iff \dfrac{d(x,y)}{1+d(x,y)} = 0 \iff d(x,y) = 0 \overset{ii}{\iff} x = y, \forall x, y \in X$

iii) $e(x,y) = \dfrac{d(x,y)}{1+d(x,y)} \overset{iii}{=} \dfrac{d(y,x)}{1+d(y,x)} = e(x,y), \forall x, y \in X$

iv) Consider the functions $f : \mathbb{R}_+ \to \mathbb{R}, f(x) = \dfrac{x}{1+x}$ and $g : \mathbb{R}_+ \to \mathbb{R}, g(x) = \dfrac{\alpha}{1+x}, \alpha \geq 0$. Because $f$ and $g$ are differentiable we can study their monotonicity by taking their first derivatives

$$\frac{\partial f}{\partial x}(x) = \frac{1(1+x) - x \cdot 1}{(1+x)^2} = \frac{1}{(1+x)^2} > 0, \forall x > 0$$

So $f$ is strictly increasing. And

$$\frac{\partial g}{\partial x}(x) = \frac{0(1+x) - \alpha \cdot 1}{(1+x)^2} = \frac{-\alpha}{(1+x)^2} \leq 0, \forall x > 0$$

So $g$ is weakly decreasing.

By subadditivity of $d$ we know that $d(x,y) \leq d(x,z) + d(z,y)$ and we have

$$\begin{aligned}
e(x,y) &= \frac{d(x,y)}{1+d(x,y)} \\
&\overset{f\ st.inc.}{\leq} \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)} \\
&= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)} \\
&\overset{g\ dec.}{\leq} \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)} \\
&= e(x,z) + e(z,y),\ \forall x,y \in X
\end{aligned}$$

We have shown that $e$ has the properties i-iv on $X$. So $e$ is a metric on $X$.

**Exercise 5**

We need to show that $\nu$ satisfies conditions i-iv on $X$, in order to prove it is a metric on $X$:

i) $\nu(x,y) = d(x,y) + e(x,y) \overset{i}{\geq} 0,\ \forall x,y \in X$ as the sum of two non-negative values.

ii) $\nu(x,y) = 0 \iff d(x,y) + e(x,y) = 0 \overset{i}{\iff} d(x,y) = 0$ and $e(x,y) = 0 \overset{ii}{\iff} x = y$, $\forall x,y \in X$

iii) $\nu(x,y) = d(x,y) + e(x,y) \overset{iii}{=} d(y,x) + e(y,x) = \nu(y,x),\ \forall x,y \in X$

iv) We have that for all $x, y, z, \in X$

$$\begin{aligned}
\nu(x,y) &= d(x,y) + e(x,y) \\
&\overset{iv}{\leq} d(x,z) + d(z,y) + e(x,z) + e(z,y) \\
&= d(x,z) + e(x,z) + d(z,y) + e(z,y) \\
&= \nu(x,z) + \nu(z,y)
\end{aligned}$$

So $\nu$ is a metric on $X$.

**Exercise 7**

We will examine the cases of the **discrete metric** ($d_\delta$) on any arbitrary set (say $X$) and **exponential distance** ($d_e$) on the real numbers, with $d_e(x,y) = |e^x - e^y|,\ \forall x,y \in \mathbb{R}$. Feel free to examine alternative distance functions (such as $d_{||}$, $d_I$, and $d_{max}$ seen in class) on appropriate carrier sets (!).

**Discrete Metric**

For all $x, y, z \in X$

$$d_\delta(x+z, y+z) = \begin{cases} 0, & x+z = y+z \\ 1, & x+z \neq y+z \end{cases}$$
$$= \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$
$$= d_\delta(x, y)$$

So $d_\delta$ is translation invariant.

**Exponential Distance**

We have that $\forall x, y, z \in \mathbb{R}$

$$\begin{aligned} d_e(x+z, y+z) &= |e^{x+z} - e^{y+z}| \\ &= |e^x e^z - e^y e^z| \\ &= |e^z(e^x - e^y)| \\ &= e^z |e^x - e^y| \\ &= e^z d_e(x, y) \end{aligned}$$

which generally does not equal $d_e(x, y)$ for every possible $z \in \mathbb{R}$. So $d_e$ is **not** translation invariant.

**Exercise 8**

The function $d_p$ is called the Minkowski distance. It is a more general metric function on $\mathbb{R}^k, k \in \mathbb{N}^*$, which covers the metrics discussed in class ($d_{||}$, $d_I$, and $d_{max}$) as special cases (for $p = 1$, $p = 2$, and as $p \to \infty$, respectively).

To prove that $d_p$ is a metric on $\mathbb{R}^k$, we need to show that it satisfies the properties of metric functions, i-iv, on $\mathbb{R}^k$. To show subadditivity we will employ Hölder's inequality.

i) For all $x, y \in \mathbb{R}^k$ and $x_i, y_i \in \mathbb{R}$ the $i$-th elements of $x$ and $y$, respectively, we have

$$\begin{aligned}
|x_i - y_i| \geq 0, \forall i \in \{1, 2, ..., k\} &\iff \\
|x_i - y_i|^p \geq 0, \forall i \in \{1, 2, ..., k\} &\iff \\
\sum_{i=1}^{k} |x_i - y_i|^p \geq 0 &\iff \\
\left( \sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}} \geq 0 &\iff \\
d_p(x, y) \geq 0 &
\end{aligned}$$

ii) For all $x, y \in \mathbb{R}^k$

$$\begin{aligned}
d_p(x, y) = 0 &\iff \\
\left( \sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}} = 0 &\iff \\
\sum_{i=1}^{k} |x_i - y_i|^p = 0 &\iff \\
|x_i - y_i|^p = 0, \forall i \in \{1, 2, ..., k\} &\iff \qquad \text{(sum of non-negative real numbers)} \\
|x_i - y_i| = 0, \forall i \in \{1, 2, ..., k\} &\iff \\
x_i = y_i, \qquad \forall i \in \{1, 2, ..., k\} &\iff \\
x = y &
\end{aligned}$$

iii) $d_p(x, y) = \left( \sum_{i=1}^{k} |x_i - y_i|^p \right)^{\frac{1}{p}} = \left( \sum_{i=1}^{k} |y_i - x_i|^p \right)^{\frac{1}{p}} = d_p(y, x), \ \forall x, y \in \mathbb{R}^k$

iv) Consider Hölder's inequality for all $x, y \in \mathbb{R}^N$

$$\sum_{i=1}^{N} |x_i y_i| \leq \left( \sum_{i=1}^{N} |x_i|^\alpha \right)^{\frac{1}{\alpha}} \left( \sum_{i=1}^{N} |y_i|^\beta \right)^{\frac{1}{\beta}}$$

with $\alpha, \beta \in (1, +\infty)$ such that $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 1$. For $\alpha = \beta = 2$ we get the Cauchy-Schwartz inequality. For $x \in \mathbb{R}^N$ we call $||x||_p := \left( \sum_{i=1}^{N} |x_i|^p \right)^{\frac{1}{p}}$ the $p$-norm of $x$.

Because of the restriction on $\alpha \neq 1$ and $\beta \neq 1$, we need to consider the case of $p = 1$ separately.

Case: $p = 1$

If for $d_p$ we have $p = 1$, then $d_p = d_{||}$ and subadditivity can be shown easily. For $p > 1$, we proceed as follows.

Case: $p > 1$

For any $x, y, z \in \mathbb{R}^k$ such that $x = y$ we want to show that

$$d_p(x,y) \leq d_p(x,z) + d_p(z,y) \overset{ii}{\iff} 0 \leq d_p(x,z) + d_p(z,y) \overset{i}{\iff} d_p(x,z) \geq 0 \text{ and } d_p(z,y) \geq 0$$

which trivially holds for all $z \in \mathbb{R}^k$.

For any $x, y, z \in \mathbb{R}^k$ such that $x \neq y$, consider the value of $d_p(x,y)$ raised to the power $p$

$$\begin{aligned}
(d_p(x,y))^p &= \sum_{i=1}^{k} |x_i - y_i|^p \\
&= \sum_{i=1}^{k} |x_i - z_i + z_i - y_i|^p \\
&= \sum_{i=1}^{k} |x_i - z_i + z_i - y_i||x_i - z_i + z_i - y_i|^{p-1} \\
&\leq \sum_{i=1}^{k} (|x_i - z_i| + |z_i - y_i|)|x_i - z_i + z_i - y_i|^{p-1} \\
&= \sum_{i=1}^{k} |x_i - z_i||x_i - z_i + z_i - y_i|^{p-1} + \sum_{i=1}^{k} |z_i - y_i||x_i - z_i + z_i - y_i|^{p-1} \\
&= \sum_{i=1}^{k} |x_i - z_i||x_i - y_i|^{p-1} + \sum_{i=1}^{k} |z_i - y_i||x_i - y_i|^{p-1}
\end{aligned}$$

We can apply Hölder's inequality for each of the two sums above. Choose $\alpha = p$ and find $\beta$ as

$$\frac{1}{p} + \frac{1}{\beta} = 1 \iff \frac{1}{\beta} = 1 - \frac{1}{p} \iff \beta = \frac{1}{1 - \dfrac{1}{p}} \iff \beta = \frac{p}{p-1}$$

So now by Hölder's inequality we have

$$\begin{aligned}
(d_p(x,y))^p &\le \left(\sum_{i=1}^{k}|x_i-z_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}\left(|x_i-y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&+ \left(\sum_{i=1}^{k}|z_i-y_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{k}\left(|x_i-y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(\left(\sum_{i=1}^{k}|x_i-z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k}|z_i-y_i|^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{k}\left(|x_i-y_i|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \\
&= \left(\left(\sum_{i=1}^{k}|x_i-z_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k}|z_i-y_i|^p\right)^{\frac{1}{p}}\right)\left(\sum_{i=1}^{k}|x_i-y_i|^p\right)^{\frac{p-1}{p}} \\
&= \left(d_p(x,z)+d_p(z,y)\right)\left(d_p(x,y)\right)^{p-1}
\end{aligned}$$

Because $x \neq y \iff d_p(x,y) \neq 0$, by multiplying both sides by $(d_p(x,y))^{1-p}$ we get

$$d_p(x,y) \le d_p(x,z) + d_p(z,y)$$

as required.

We have proven that the Minkowski distance, $d_p$, satisfies conditions i-iv on $\mathbb{R}^k, k \in \mathbb{N}^*$, so $d_p$ is a metric on $\mathbb{R}^k, k \in \mathbb{N}^*$.

**Exercise 12**

To define a metric, a carrier set is needed, which the exercise does not provide. So the answer is no.

If we were to choose as a carrier the set $X \subseteq \mathbb{R}^n, n \in \mathbb{N}$, we can show that the proposed distance function (henceforth $d$) does not possess the properties of metric functions (even if $A$ a positive definite matrix). For example:

$X = \mathbb{R}^2, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$d(x,y) = (x-z)'A(x-z) = (0-2)^2 + (0-0)^2 = 4$

$d(x,z) = (x-y)'A(x-y) = (0-1)^2 + (0-0)^2 = 1$

$d(z,y) = (z-y)'A(z-y) = (1-2)^2 + (0-0)^2 = 1$

which give $d(x,y) > d(x,z) + d(z,y)$ (because $4 > 1+1$), which violates the triangle inequality.