Exercises

1. Suppose that $X=\{(x_n)_{n\in\mathbb{N}},\ x_n\in\mathbb{R},\ \forall n\in\mathbb{N},\ \sum_{i=0}^{\infty} x_i^2<+\infty\}$, $d(x,y):=\sum_{i=0}^{\infty}(x_i-y_i)^2$ for any $x=(x_n)_{n\in\mathbb{N}},\ y=(y_n)_{n\in\mathbb{N}}\in X$. Consider $O_d[0,1]$ and show that it is not d-totally bounded. $\square$

2. Suppose that $X=\{f:[a,b]\to\mathbb{R}:\ \int_a^b f^2(x)dx<+\infty\}$ and $d(f,g):=$
$=\int_a^b (f(x)-g(x))^2dx$. For $0:[a,b]\to\mathbb{R}$, $0(x):=0\ \forall x\in[a,b]$, consider $O_d[0,1]$ and show that it is not d-totally bounded. $\square$

3. For (X,d) a general metric space, and $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$, $x_n,y_n,x,y\in X$ $\forall n\in\mathbb{N}$, with $x=d\text{-}\lim(x_n)$ and $y=d\text{-}\lim(y_n)$. Prove that $d(x_n,y_n)\to d(x,y)$ w.r.t. the usual metric on $\mathbb{R}$. Conclude that $d:X\times X\to\mathbb{R}$ is appropriately continuous.

4. For the framework of exercise 3, show that $d(\cdot,x):X\to\mathbb{R}$ is d_I/d-continuous where d_I is the usual metric on $\mathbb{R}$. Conclude analogously for $d(x,\cdot)$. $\square$

5. Show that $\tau_d:=\{A\subseteq X,\ A \text{ is } d\text{-open}\}$ satisfies the axioms of a topology, i.e.

a. $\emptyset, X\in\tau_d$

b. for I an index set, $A_i\in\tau_d,\ \forall i\in I\Rightarrow\bigcup_{i\in I}A_i\in\tau_d$.

c. for I a finite index set, $A_i\in\tau_d,\ \forall i\in I\Rightarrow\bigcap_{i\in I}A_i\in\tau_d$.

6. In the framework of exercise 6, show that $\tau_d':=\{A\subseteq X,\ A \text{ is } d\text{-closed}\}$ satisfies the dual to the above axioms i.e.,

a'. $\emptyset, X\in\tau_d'$,

b'. if I is a finite index set, $A_i\in\tau_d',\ \forall i\in I\Rightarrow\bigcup_{i\in I}A_i\in\tau_d'$

c'. if I is an index set, $A_i\in\tau_d',\ \forall i\in I\Rightarrow\bigcap_{i\in I}A_i\in\tau_d'$.

7. If $A\subseteq X$, $\operatorname{int}A := \{x\in A : \exists \varepsilon_x>0 : O_d(x,\varepsilon_x)\subseteq A\}$. Prove that $A\in\tau_d$ iff $A=\operatorname{int}A$ (i.e. the d-open sets are exactly the d points of this interior operator).

8. If $A\subseteq X$ then $\operatorname{ext}A := \{x\in A' : \exists \varepsilon_x>0 : O_d(x,\varepsilon_x)\cap A=\emptyset\}$. Show that $\operatorname{ext}A\in\tau_d$, $\forall A\subseteq X$. Show that $A\in\tau_d'$ iff $\operatorname{ext}A=A'$. $\square$

9. If $A\subseteq X$ then $\operatorname{bd}A=\{x\in X : \forall\varepsilon>0,\ O_d(x,\varepsilon)\cap\operatorname{int}A\neq\emptyset$ and $O_d(x,\varepsilon)\cap\operatorname{ext}A\neq\emptyset\}$. Show that $\operatorname{bd}A\in\tau_d'$, $\forall A\subseteq X$. Show that $A\in\tau_d' \Leftrightarrow \operatorname{bd}A\subseteq A$. Show that $A\in\tau_d$ iff $A\cap\operatorname{bd}A=\emptyset$. $\square$

10. Show that continuity is preserved by composition. $\square$

11. Show that f is d_Y/d_X-continuous iff $\forall C\in\tau_{d_Y}'$, $f^{-1}(C)\in\tau_{d_X}'$. $\square$

12. Show that every singleton set in any metric space is closed. However if $x\in\mathbb{R}^k$, then $\operatorname{int}\{x\}=\emptyset$ w.r.t. d_I. $\square$

[The notes are in a state of perpetual correction. They do not substitute the lectures. Please report any typos to stelios@aueb.gr or the course's e-class.]