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Why the distinction between Point-wise and Uniform convergence may matter a lot: An economic example.

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Consider the following real-world situation: a new production process is introduced in a factory. Experience has taught us that there exist a "learning curve", a gradual increase in efficiency related to the implementation of a new production process and the output level obtained by it. Let $z \in (0,1]$ represent the "intensity" with which we learn how to manage and implement efficiently the new process. We excluded 0 from the domain to reflect the fact (or the belief) that we always pay even a little bit of attention. Let $Y_e = F\left(\bar{K}, \bar{L}\right) > 0$ represent the *full-efficiency* level of output, for given pre-specified levels of capital and labor used. Let n denote the number of times the process runs a full cycle. Assume that actual output, expressed as a function of learning intensity is

$$f_n(z) = \left(1 - \frac{1}{a + (n-1)z}\right) Y_e, \quad a > 1$$

The term multiplying Y_e is the one representing the "learning-curve" effect, the fact that we are only gradually approaching full efficiency, based on the number n of times we have run the process (this is Kenneth Arrow's "learning-by-doing" concept), but also on the intensity z with which we learn its nuts and bolts (a conscious effort to increase efficiency). Note the interaction between them, which is a realistic assumption. The constant a determines the initial level of efficiency that will materialize in the first-ever production cycle ($n = 1$), based on ability, knowledge and past experience. We have suppressed the presence of production inputs since we consider the case where they are fixed throughout.

The production function sequence ***converges pointwise*** to the limit $f = Y_e$ for any given *fixed* $z \in (0,1]$ and this is elementary to show. **In economic terms, this is translated: with *any*, even a little, but *fixed* (constant, time-invariant) level of learning intensity, we will eventually reach full-efficiency output.** That's comforting, even though the speed at which we will approach full efficiency may not be satisfactory if z is fixed at "too low" a level (but that is a different consideration).

But the function does **not** converge uniformly. To obtain uniform convergence it must be the case that

$$\forall \varepsilon > 0,\ \exists\, n_0(\varepsilon): \left| f_n(z) - f \right| < \varepsilon \quad \forall n \geq n_0(\varepsilon),\ \forall z \in (0,1]$$

Ad absurdum, assume that this holds, so we must have

$$\left| f_{n_0}(z) - f \right| = \left| \left(1 - \frac{1}{a + (n_0 - 1)z} \right) Y_e - Y_e \right| = Y_e \frac{1}{a + (n_0 - 1)z} < \varepsilon, \quad \forall \varepsilon > 0\ \ \forall z \in (0,1]$$

Rearranging we have that then it should be the case that $z > \dfrac{(Y_e/\varepsilon) - a}{(n_0 - 1)}$

The numerator of the right-hand-side will certainly be strictly positive, for sufficiently small values of ε. As for the denominator, although n_0 can be a very large number, it will always be a finite natural number. So we conclude the right-hand-side will be strictly greater than zero, which bounds z away from zero, and so violates the requirement that the relation holds $\forall z \in (0,1]$. So the sequence does not converge uniformly.

From another angle, note that an equivalent condition for uniform convergence is that

$$\sup_z \left| f_n(z) - f \right| \to 0, \quad n \to \infty$$

Now, if for $n > 1$ we set $z = 1/(n-1)$ we get $\left| f_n(z) - f \right|_{z=1/(n-1)} = Y_e \dfrac{1}{a+1} > 0,\ \forall n$.

But also we have that $\left|f_n(z)-f\right|_{z=1/(n-1)} < \sup_z\left|f_n(z)-f\right|$ (**show it**). Combining, we get $0<\left|f_n(z)-f\right|_{z=1/(n-1)} < \sup_z\left|f_n(z)-f\right|, \quad \forall n$

i.e. by having z getting smaller and smaller in value in this way, we have bounded the supremum away from zero $\forall n$, showing thus that uniform convergence does not hold (note that by setting $z=1/(n-1)$ we essentially make the elements of the function sequence *constants* and so the sequence has trivially a limit -but not the one we want it to have!)

So our output sequence converges pointwise but not uniformly. Do we care? Does the fact that we have only pointwise but not uniform convergence reflects here something real-world important?

It does: it reflects the fact that, to achieve full-efficiency, or even to just see efficiency increase, ***we cannot rely on "automatic learning-by-doing"*** (the increase in n), because any gains through this channel may be offset by losses in learning intensity. It therefore reflects the fact that we have to monitor and manage the "learning intensity" of the people working in the factory, and at least try to stabilize it (to exploit the property of pointwise convergence). Otherwise we run the risk of seeing our production process, after perhaps an initial period where it exhibits increasing efficiency, ending up either stalling at a below full-efficiency level, or even regressing... and loss of potential or actual efficiency is one of the most important issues in real-world economies.

Informal discussion

We examine the "journey" of a function looking at two influencing factors (n and z). In examining **pointwise convergence**, we "commit" one of them, z, to a specific value beforehand, so we examine *separate* scenarios of a function that is now influenced only by a single factor. In an informal sense, we have transformed a bivariate function to many univariate ones. We still require that each and everyone of these univariate functions converges, but nevertheless, this is not as "difficult" to obtain as when having two influencing factors varying at the same time...

...which is exactly what we do in examining **uniform convergence**. Here we allow for a possible "interplay" between n and z, that may not permit the function sequence to converge to the limit that it converges pointwise.--