# Total Boundedness and a Uniform Law of Large Numbers

Suppose that $X_1, X_2, \dots, X_n$ are iid random variables (with values in $\mathbb{R}$), and $\mathcal{A}$ is a $d_{sup}$- totally bounded non-empty subset of $B(\mathbb{R}, \mathbb{R})$

- Pointwise Law of Large Numbers

$\forall f \in \mathcal{A}$, $\forall \varepsilon > 0$,

$$(*) \quad \lim_{n\to+\infty} \mathbb{P}\left( \left| \frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}[f(X_0)] \right| > \varepsilon \right) = 0$$

Proof. Exercise! Use the fact that $\mathcal{A}$ is d-sup totally bounded, and hence uniformly bounded (why?) to prove that $\forall f \in \mathcal{A}$, $\mathbb{E}\,|f(X_0)|^2 < +\infty$, and then use the iid assumption with Markov's inequality. ∎

Question: Can $(*)$ be extended to hold uniformly on $\mathcal{A}$, i.e. to show that, $\forall \varepsilon > 0$

$$\lim_{n\to+\infty} \mathbb{P}\left( \sup_{f\in\mathcal{A}} \left| \frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_0)) \right| > \varepsilon \right) = 0 \quad ?$$

Total boundedness of $\mathcal{A}$ becomes handy:

a. Notice that if $J \in \mathbb{N}^*$, $f_1, f_2, \dots, f_J \in \mathcal{A}$, then $(*) \Rightarrow$ why?

$$\forall \varepsilon > 0, \lim_{n\to+\infty} \mathbb{P}\left( \sum_{j=1}^{J} \left| \frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0)) \right| > \varepsilon \right) = 0 \quad (**)$$

b. total boundedness implies that: $\forall \delta > 0$, $\exists J(\delta) \in \mathbb{N}^*$ and $f_1, f_2, \dots, f_{J(\delta)} \in A$, and $\bigcup_{j=1}^{J(\delta)} O_{d_{sup}}[f_j, \delta] \supseteq A$. (*)

c. Fix $\delta > 0$ and notice that: for any $\epsilon > 0$
↳ to be chosen later on

$$\mathbb{P}\left(\sup_{f \in A} \left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_0))\right| > \epsilon\right) \overset{(*)}{\leq}$$

$$\mathbb{P}\left(\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_0))\right| > \epsilon\right) \leq$$

↳ $\pm f_j(X_i)$ ↳ $\pm \mathbb{E}(f_j(X_0))$

by tr. ineq. and monotonicity of prob.

$$\mathbb{P}\left(\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left|\frac{1}{n}\sum_{i=1}^{n}\left(f(X_i) - f_j(X_i)\right) + \frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0)) + \mathbb{E}(f_j(X_0)) - \mathbb{E}(f(X_0))\right| > \epsilon\right)$$

$\leq$ tr. ineq. + mon. of $\mathbb{P}$

$$\mathbb{P}\left(\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left[\left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - f_j(X_i)\right| + \left|\frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0))\right| + \left|\mathbb{E}(f_j(X_0)) - \mathbb{E}(f(X_0))\right| > \epsilon\right)$$

$\leq$ subadd. of max sup + mon of $\mathbb{P}$

$$\mathbb{P}\Bigg(\underbrace{\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - f_j(X_i)\right|}_{A} + \underbrace{\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left|\frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0))\right|}_{B}$$

$$+ \underbrace{\max_{j=1,\dots,J(\delta)} \sup_{f \in O_{d_{sup}}[f_j,\delta]} \left|\mathbb{E}(f_j(X_0)) - \mathbb{E}(f(X_0))\right|}_{C} > \epsilon\Bigg) \quad (***)$$

d. Notice that

① $$A \leq \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \frac{1}{n}\sum_{i=1}^{n} |f(x_i) - f_j(x_i)|$$

tr.ineq
mon of
maxsup

$$\leq \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \frac{1}{n}\sum_{i=1}^{n} \sup_{x\in\mathbb{R}} |f(x) - f_j(x)|$$

mon of
maxsup

$$= \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \frac{n}{n} d_{sup}(f, f_j) = \delta \text{ (why?)}$$

② $$B \overset{(**)}{\leq} \sum_{j=1}^{J(\delta)} \left| \frac{1}{n}\sum_{i=1}^{n} f_j(x_i) - \mathbb{E}(f_j(X_0)) \right|$$

③ $$C \leq \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \mathbb{E}\left(| f_j(X_0) - f(X_0)|\right)$$

lin.
of int

$$\leq \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \mathbb{E}\left(\sup_{x\in\mathbb{R}} | f_j(x) - f(x) |\right)$$

Mon.
of int

$$= \max_{j=1,\dots,J(\delta)} \sup_{f \in \mathcal{O}_{d_{sup}}[f_j,\delta]} \mathbb{E}\left(d_{sup}(f_j, f)\right)$$

$$\leq \delta$$

e. Hence by (1), (2), (3) (***) is less than or equal to

$$\mathbb{P}\left(\sum_{j=1}^{J(\delta)} \left|\frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0))\right| > \epsilon - 2\delta\right)$$

f. Thereby, for any $\epsilon > 0$, choosing $\delta < \frac{\epsilon}{3}$ and using (**) we have that

why does this hold for limits?

$$\lim_{n\to+\infty} \mathbb{P}\left(\sup_{f\in A} \left|\frac{1}{n}\sum_{i=1}^{n} f(X_i) - \mathbb{E}(f(X_0))\right| > \epsilon\right)$$

$$\leq \lim_{n\to+\infty} \mathbb{P}\left(\sum_{j=1}^{J(\delta)} \left|\frac{1}{n}\sum_{i=1}^{n} f_j(X_i) - \mathbb{E}(f_j(X_0))\right| > \epsilon - 2\delta\right)$$

$$= 0$$

And thereby the first limit is zero and the ULLN is proven (why?) ☒

Exercise: let $A = \{\sin(\theta x),\ \theta \in [0,1]\}$.
Do the above hold for this A?