Uniform Limit of Self-function Autocomposition

Suppose that $f: X \to X$ is a self function, and for $n \in \mathbb{N}$ define

$$f^{(n)} := \begin{cases} id_X & , n=0 \\ \underbrace{f \circ f \circ \ldots \circ f}_{n\text{-fold}} & , n>0 \end{cases}.$$

How does $f^{(n)}$ behaves as $n \to +\infty$? With somewhat more structure to this framework, $f^{(n)}$ uniformly converges to a special constant function on X:

a. X is endowed with a metric d.

b. $B(X,X)$ denotes the set of bounded self-functions on X, i.e. $f \in B(X,X)$ iff $\exists y \in X, \varepsilon > 0: f(X) \subseteq O_d(y,\varepsilon)$.

Notice that since $X \neq \emptyset$, $B(X,X) \neq \emptyset$ since if $f(x) = x^* \in X$ (i.e. it is a constant self function), $f \in B(X,X)$ (why?). In what follows a constant self-function on X will be identified with its constant value for notational simplicity.

c. If (X,d) is itself bounded, then every self-function on X belongs to $B(X,X)$ (why?).

d. As we know, we can endow $B(X,X)$ with the uniform metric stemming from d, i.e. $f,g \in B(X,X)$, $d^X_{sup}(f,g) := \sup_{x \in X} d(f(x), g(x))$, and study uniform convergence within $B(X,X)$.

We thus obtain the following result:

Lemma. Suppose that i. (X,d) is bounded, ii. (X,d) is complete, iii. f is a d-contraction. Then as $n \to +\infty$, $d^X_{sup}(f^{(n)}, x_f) \to 0$, where x_f denotes the constant function at the unique fixed point of f.

Proof. ii + iii + BFPT imply that x_f is well-defined and b. above that $x_f \in B(X,X)$.