Another Asymptotic Comparison

Suppose that d, d^* are well defined metrics with which X can be equipped and furthermore that,

$$\forall x \in X, \exists n(x): \mathcal{O}_d(x,n) \subseteq \mathcal{O}_{d^*}(x,n), \forall n \geq n(x), \quad [*]$$

i.e. the relevant open balls satisfy a set of asymptotic inclusion restrictions.

[*] implies that if A is d-bounded then it is also d^*-bounded,

and this is due to that if $\exists x \in X, \varepsilon > 0: A \subseteq \mathcal{O}_d(x,\varepsilon)$ then for $n(\varepsilon) :=$ the smallest natural number greater than or equal to ε and $\mu := \max(n(\varepsilon), n(x))$ we also have that (why?) $A \subseteq \mathcal{O}_d(x,\mu) \subseteq \mathcal{O}_{d^*}(x,\mu)$.

Exercise. Does [*] imply that if A is d-totally bounded then it is also d^*-totally bounded?

Notice that [*] also implies the following (in a rather trivial manner): if $x, y_n \in X$ and such that $d(x,y_n) \to +\infty$ as $n \to +\infty$, then

$$\frac{d^*(x,y_n)}{d(x,y_n)} \longrightarrow c_x,$$

where $c_x \in (0,1]$ and thereby $d^*(x,\cdot) = O(d(x,\cdot))$, or $c_x = 0$ and thereby $d^*(x,\cdot) = o(d(x,\cdot))$.