Boundness Can Be "Asymptotically Lost"

Suppose that $X=\mathbb{R}$ and for any $n\in\mathbb{N}^*$ consider

$$d_n(x,y) = \begin{cases} |x-y|, & \text{if } |x-y|<n \\ n, & \text{if } |x-y|\geq n \end{cases}, \text{ for any } x,y\in\mathbb{R}.$$

Show that d_n is a well-defined metric (you essentially have to show that the triangle inequality holds). Notice that as $n\to+\infty$, for any $x,y\in\mathbb{R}$, $|x-y|<n$ eventually, hence pointwise (i.e. $\forall x,y\in\mathbb{R}$)

$$d_n \to d_u.$$

Notice however that despite this convergence, d_n-boundness may be asymptotically lost in the following sense:

First it is easy to see that $O_{d_n}(x,\varepsilon) = \begin{cases} O_{d_u}(x,\varepsilon), & \varepsilon\leq n \\ \mathbb{R}, & \varepsilon>n \end{cases}.$

Then, e.g. for $A=\mathbb{R}$ we have that $\forall n\in\mathbb{N}^*$, $\mathbb{R}$ is d_n-bounded yet $\mathbb{R}$ is not d_u-bounded and thereby this property is not retained in the limit, hence it is not in some sense a "continuous" property.